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ABSTRACT

This is part two of a two-part SMSG intermediate mathematics text for high school students. The aim of the text is to focus attention on mathematical ideas which are appropriate for study by college-capable students in the eleventh grade. Chapter topics include number systems, an introduction to coordinate geometry in the plane, the function concept and the linear function, quadratic functions and equations, complex number systems, equations of the first and second degree in two variables, systems of equations in two variables, and systems of first degree equations in three variables. (MP)

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STUDENT'S TEXT

UNIT NO. 18

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INTERMEDIATE MATHEMATICS

PART II

SE 027 894

SCHOOL MATHEMATICS STUDY GROUP

Intermediate Mathematics

Student's Text, Part II

REVISED EDITION

Prepared under the supervision of
the Panel on Sample Textbooks
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Chapter 9

LOGARITHMS AND EXPONENTS

9-1. *A New Function*: $y = \log x$. Consider the shaded rectangle in Fig. 9-1a; it is bounded by the x-axis, the line $y = 2$, and the vertical lines erected at $x = 2$ and at the point whose abscissa is x. If $x > 2$, the length of the base of the rectangle is $x - 2$; if $x < 2$, then $x - 2$ is the negative of the length of the base. In all cases the altitude of the rectangle is 2. Thus $2(x - 2)$, or $2x - 4$, is an expression whose value is the area of the rectangle if $x > 2$, whose value is 0 if $x = 2$, and whose value is the negative of the area of the rectangle if $x < 2$. Set $y = 2x - 4$. The shaded rectangle has been used to define a correspondence between x and y which is a linear function. The graph of $y = 2x - 4$ is shown in Fig. 9-1b.

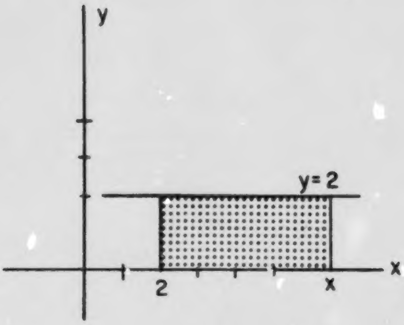

Figure 9-1a. The shaded region is used to define a linear function.

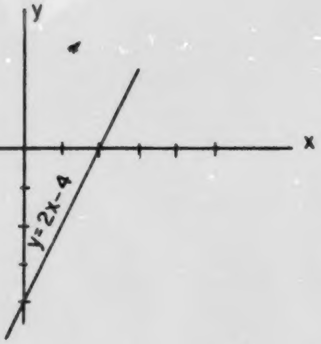

Figure 9-1b. Graph of $y = 2x - 4$.

Consider the shaded trapezoid in Fig. 9-1c; it is bounded by the x-axis, the line $y = -2x + 4$, and the vertical lines erected at $x = 1$ and at the point whose abscissa is x.

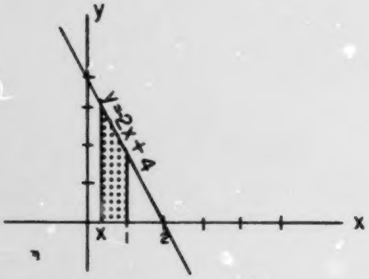

Figure 9-1c. The shaded region is used to define a quadratic function.

For the purposes of this illustration only those values of x in the interval $0 \leq x \leq 2$ will be considered. If $1 < x \leq 2$, the length of the altitude of the trapezoid is $x - 1$; if $0 \leq x < 1$, then $x - 1$ is the negative of the length of the altitude. The base of the trapezoid at $x = 1$ is 2, and the base at x is $(-2x + 4)$. One-half the sum of these bases is $\frac{(-2x + 4) + 2}{2}$ or $(-x + 3)$. Since the area of a trapezoid is the product of the altitude and one-half the sum of the bases, we have $(-x + 3)(x-1) = -x^2 + 4x - 3$ as the expression whose value is the area of the trapezoid if $1 < x \leq 2$, whose value is 0 if $x = 1$, and whose value is the negative of the area of the trapezoid if $0 \leq x < 1$.

Set $y = -x^2 + 4x - 3$. The shaded trapezoid has been used to define a correspondence between x and y which is a quadratic function. The graph of $y = -x^2 + 4x - 3$ is shown in Fig. 9-1d.

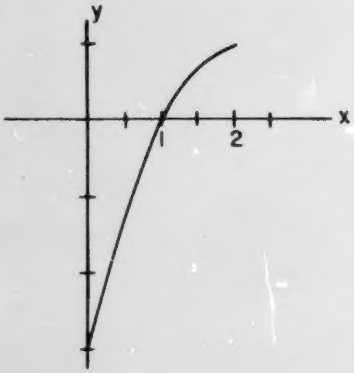

Figure 9-1d. Graph of $y = -x^2 + 4x - 3$ for $0 \leq x \leq 2$.

The functions obtained in the two foregoing examples are the familiar linear and quadratic functions which were studied in Chapters 3 and 4. Other functions can be obtained by considering the areas under other curves, and some of these functions are entirely new and unfamiliar.

Consider the shaded region in Fig. 9-1e; it is bounded by the x-axis, the hyperbola $y = k/x$, and the vertical lines erected at $x = 1$ and at the point whose abscissa is x. Restrict x arbitrarily to have only values greater than zero.

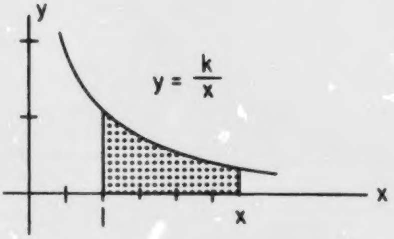

Figure 9-1e. The shaded region is used to define the logarithm of x.

There is no simple formula that gives the area of the shaded region; however, the shaded region will be used to define a function just as in the two previous examples. The new function is known as the logarithm of x; it is denoted by $y = \log x$.

The following definition describes the logarithm function as a correspondence between x and y.

Definition 9-1. The logarithm function is defined for all $x > 0$ by the following correspondence between x and y.

(a) For each $x > 1$, the corresponding value of y is the area of the region bounded by the x-axis, the hyperbola $y = k/x$, and the vertical lines at 1 and x.

(b) For $x = 1$, the value of y is 0.

(c) For each x such that $0 < x < 1$, the value of y is the negative of the area bounded by the x-axis, the hyperbola $y = k/x$, and the vertical lines at 1 and at x.

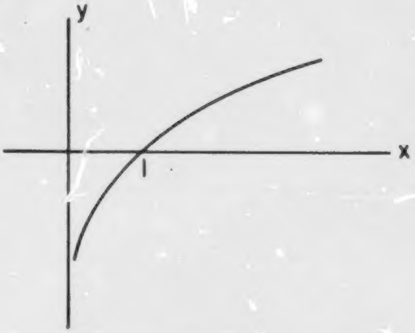

Figure 9-1f. Graph of $y = \log x$.

Fig. 9-1f. shows the graph of $y = \log x$. It follows from the definition that the graph lies below the x-axis for $0 < x < 1$, crosses the x-axis at $x = 1$, and lies above the x-axis for $x > 1$. Furthermore, the curve rises as x increases.

For each fixed value of k, $y = k/x$ is a hyperbola and the correspondence described in Definition 9-1 defines a logarithm function. Thus it is clear that a logarithm function can be defined for each fixed value of k. However, in this course only those logarithm functions which arise from positive values of k will be considered. Fig. 9-1e shows a hyperbola for $k = 1$, and Fig. 9-1f shows the graph of the corresponding logarithm function.

The properties of all logarithm functions will be derived simultaneously. Two of the logarithm functions are especially important in mathematics and in applications to other subjects. If $k = 1$, the corresponding logarithm function is known as the natural logarithm function. It is denoted by $y = \ln x$. The natural logarithm function has special properties which make it useful in theoretical work in mathematics and its applications. Another special choice of k gives a logarithm function whose value is 1 at $x = 10$. This logarithm function is called the common logarithm function. It is denoted by $y = \log_{10} x$ The common logarithm function is exceedingly useful in numerical calculations, due to the fact that our number system makes use of base ten. Tables of values of natural logarithms and common logarithms are included in most standard books of tables.

Fig. 9-1g shows the graph of $y = 1/x$ from $x = 0.65$ to $x = 1.00$, and Fig. 9-1h shows the graph of the same hyperbola from $x = 1.00$ to $x = 1.35$. These figures can be used to compute a rather accurate table of natural logarithms. Observe first of all that each small square in these figures has an area of 0.0001. A good approximation to the area under the hyperbola can be obtained by counting squares. As an example, compute $\ln 1.05$. Fig. 9-1h shows that the number of whole squares under the curve from $x = 1$ to $x = 1.05$ is 485; thus, $\ln 1.05$ is approximately 0.0485. A more accurate value can be obtained by adding on the areas of the parts of squares that lie under the hyperbola. In this case the curve crosses five squares, and the graph indicates that the parts of their areas that lie under the hyperbola amount to slightly more than two whole squares. Thus, a more accurate value for $\ln 1.05$ is 0.0487. Tables show that the value of $\ln 1.05$ correct to five decimal places is 0.04879. (See Mathematical Table from Handbook of Physics and Chemistry.) A similar calculation for Fig. 9-1g shows that the approximate value of $\ln 0.95$ is -0.0513; the value given by five-place tables is -0.05129.

The graph of $y = \ln x$ is shown in Fig. 9-1i. It contains the two points $(0.95, -0.0513)$ and $(1.05, 0.0487)$ whose coordinates were computed from Figs. 9-1g and 9-1h. It is obviously necessary to extend the graph of $y = 1/x$ both to the left and to the right in order to obtain enough points to plot the graph of $y = \ln x$ shown in Fig. 9-1i.

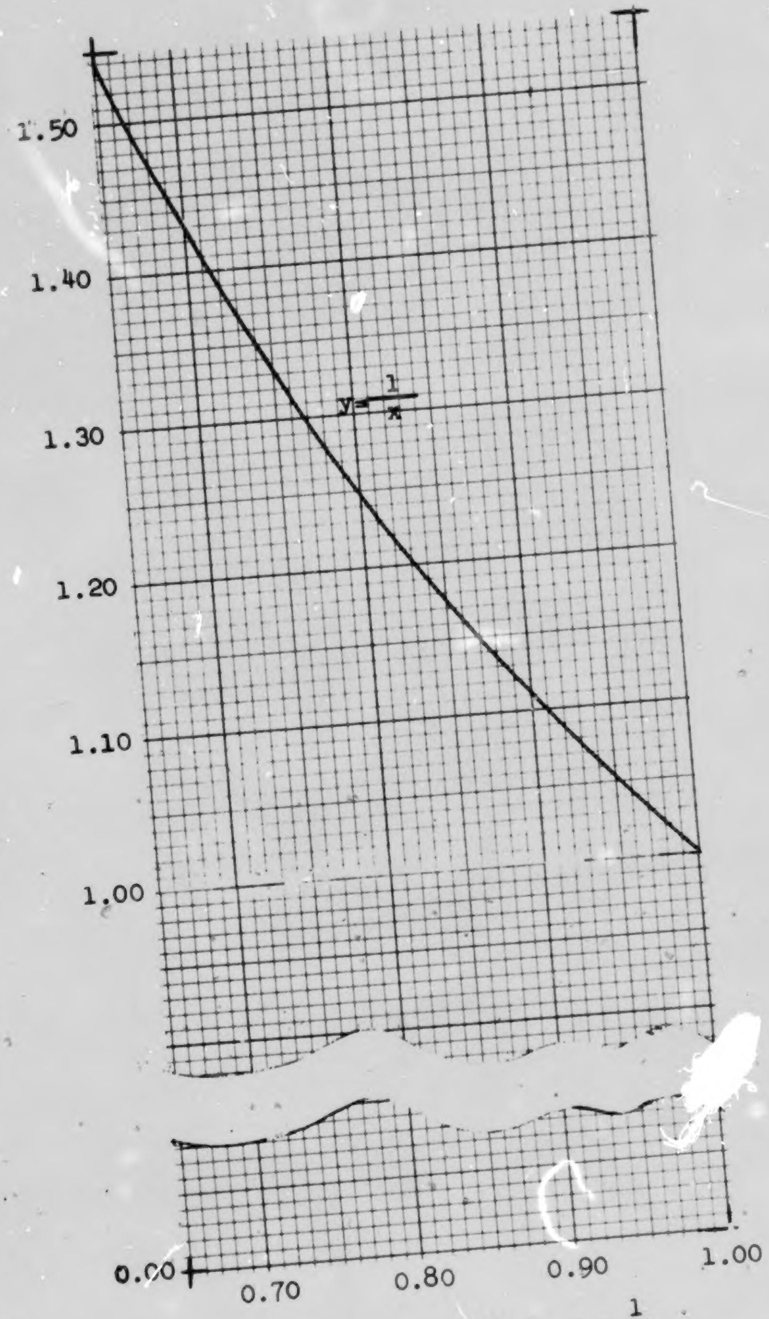

Figure 9-1g. Gra[illegible] of $y = \frac{1}{x}$.

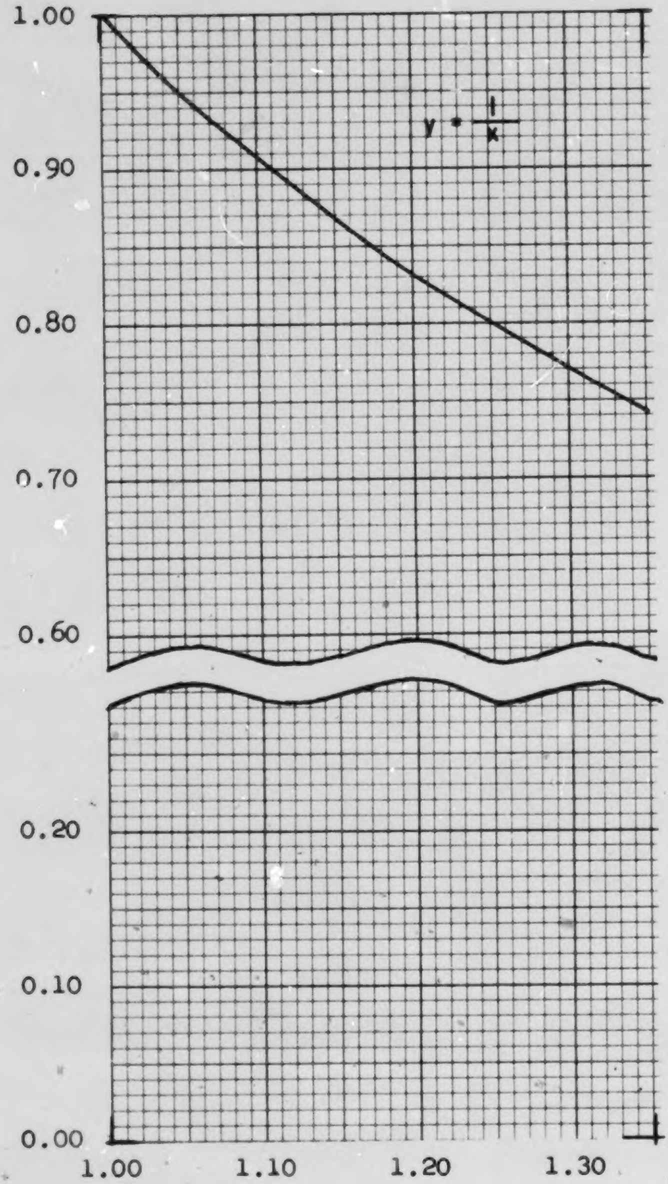

Figure 9-1h. Graph of $y = \frac{1}{x}$.

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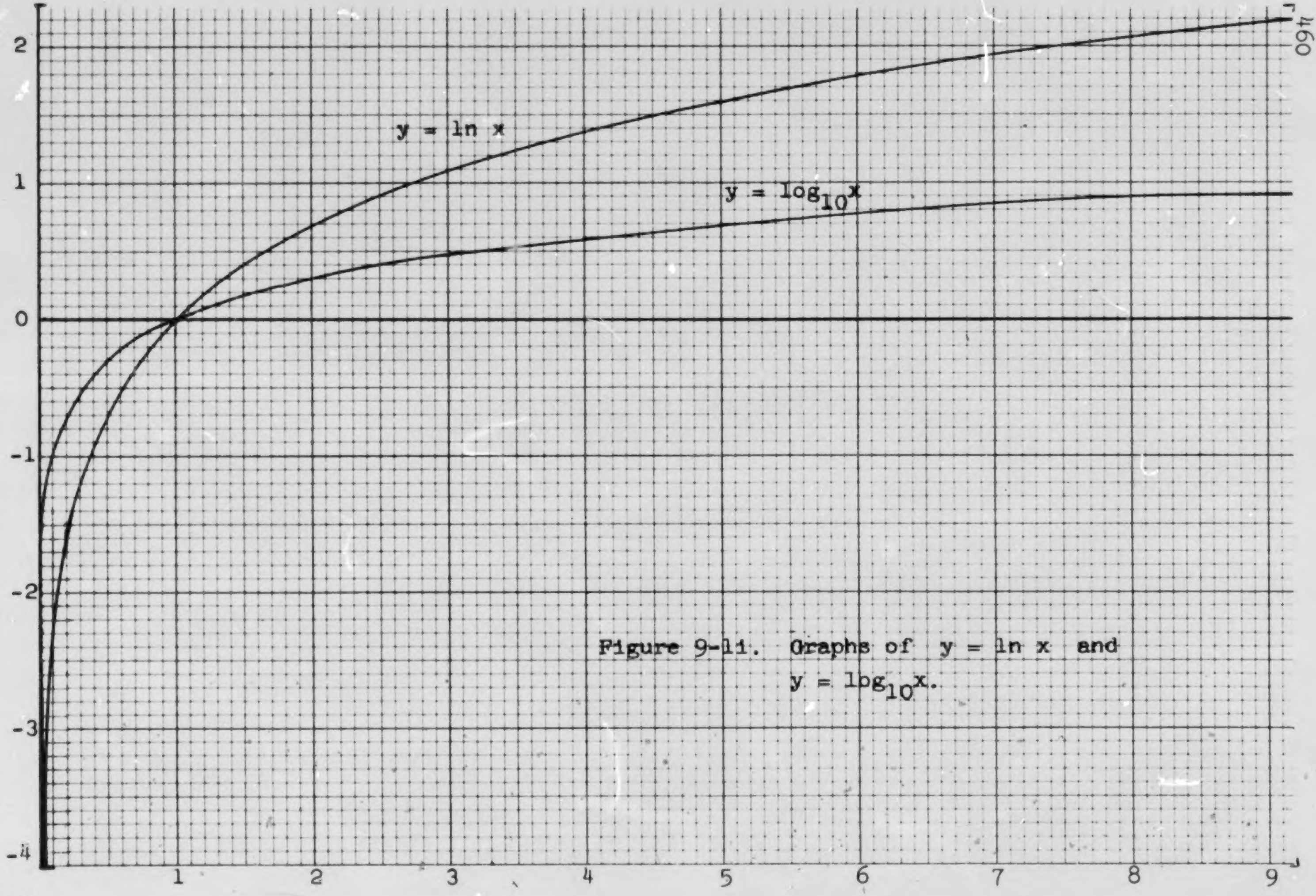

Figure 9-11. Graphs of $y = \ln x$ and $y = \log_{10} x$.

It can be shown by counting squares that ln 10 is approximately 2.3 (a five-place table gives this value as 2.30259). Experience has shown that one of the most useful logarithms is the common logarithm $\log_{10}x$; its value is 1 at $x = 10$. To show how points needed to draw the graph of $y = \log_{10}x$ can be obtained let us consider once more the hyperbola $y = k/x$. (See Fig. 9-1j). Every ordinate on the graph of this equation is k times the corresponding ordinate on the graph of $y = \frac{1}{x}$. (Notice that the height at each x for $y = \ln x$ is approximately 2.3 times the height of the ordinate for the graph $y = \log_{10}x$).

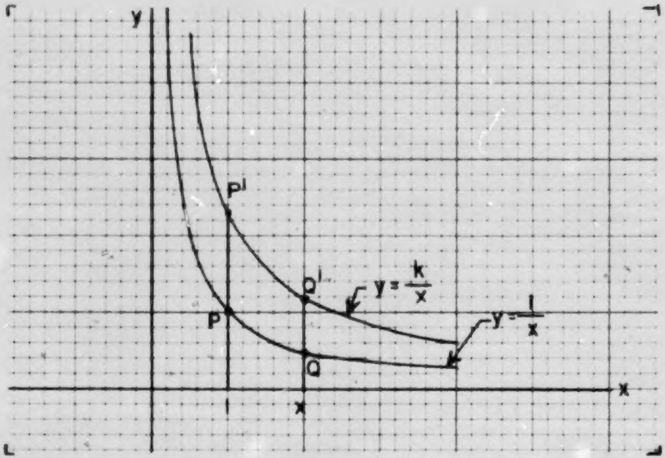

Figure 9-1j

Therefore, for any interval 1 to x $(x > 0)$, the area under the graph of $y = \frac{k}{x}$ is k times the corresponding area under the graph of $y = \frac{1}{x}$. In Fig. 9-1j this means that the area under arc P'Q' is k times the area under are PQ.

But, by our definition, these areas are $\log x$ and $\ln x$ respectively, therefore we can write

9-1 $$\log x = k \ln x,$$

where k is the constant in the equation of the hyperbola used in defining $\log x$. Clearly, the value of $\log x$ depends on k. To find an approximate value of k which makes $\log 10$ equal to one, we substitute $x = 10$ in (9-1) and make use of the fact that $\ln 10$ is approximately 2.3 as shown in Fig. 9-1i:

$$1 \approx k \times 2.3 \quad \text{or} \quad k \approx \frac{1}{2.3}.$$

Of course, $\frac{1}{2.30259}$ is a better approximation because it is based on the more accurate value of $\ln 10$ given above. The exact value is

$$\frac{1}{\ln 10}.$$

This number is denoted by M; it is an irrational number (similar to π). Its value, correct to 20 decimal places, is

9-1a $$M \approx 0.43429\ 44819\ 03251\ 82765.$$

The common logarithm of x, denoted by $\log_{10}x$, is thus the area under the hyperbola

9-1b $$y = \frac{M}{x}$$

from 1 to x. The values of $\log_{10}x$ can be computed in the same way that the values of $\ln x$ were computed. Fig. 9-1k contains the graph of $y = M/x$ from $x = 1.00$ to $x = 1.35$. Areas under this curve can be computed by counting squares. The graph of $y = \log_{10}x$ is shown in Fig. 9-1i, and Table 9-1 contains a brief table of values of $\log_{10}x$. It follows from the definition of common logarithms that

9-1c $$\log_{10}x = M \ln x.$$

As a matter of notation, $y = \log x$ will be used to denote the general logarithm function obtained from the hyperbola $y = k/x$, where the value of k is general and unspecified except that $k > 0$.

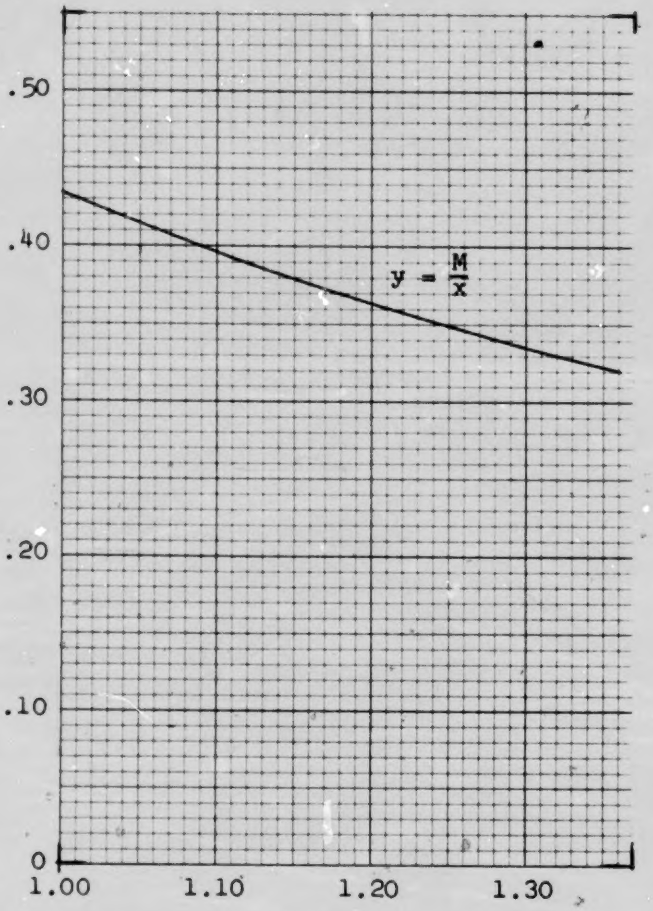

Figure 9-1k. Graph of $y = \frac{M}{x}$

Table 9-1. A Brief Table of Common Logarithms.

x	$\log_{10}x$
0.00001	-5
0.0001	-4
0.001	-3
0.01	-2
0.10	-1
1.00	0.0000
1.50	0.1761
2.00	0.3010
2.50	0.3979
3.00	0.4771
3.50	0.5441
4.00	0.6021
4.50	0.6532
5.00	0.6990
5.50	0.7404
6.00	0.7782
6.50	0.8129
7.00	0.8451
7.50	0.8751
8.00	0.9031

x	$\log_{10}x$
8.50	0.9294
9.00	0.9542
9.50	0.9777
10.00	1.0000
10.50	1.0212
11.00	1.0414
12.00	1.0792
13.00	1.1139
14.00	1.1461
15.00	1.1761
16.00	1.2041
17.00	1.2304
18.00	1.2553
19.00	1.2788
20.00	1.3010
25.00	1.3979
30.00	1.4771
35.00	1.5441
40.00	1.6021
45.00	1.6532
50.00	1.6990

Exercises 9-1

1. Use the graphs in Figs. 9-1g and 9-1h to estimate the value of ln x for those values of x listed in the first column of the following table. Compare your estimated values with the correct values given in the last column. It should be observed that these logarithms are natural logarithms rather than the common logarithms given in Table 9-1.

x	Estimated ln x	Correct ln x
0.70		-0.35667
0.82		-0.19845
0.90		-0.10536
1.12		0.11333
1.18		0.16551
1.23		0.20701
1.24		0.21511
1.26		0.23111
1.28		0.24686
1.29		0.25464
1.31		0.27003
1.32		0.27763
1.33		0.28518
1.34		0.29267

2. Use the graph in Fig. 9-1k to estimate the values of $\log_{10}x$ for those values of x listed in the first column of the following table. Compare your estimated values with the correct values given in the last column.

x	Estimated $\log_{10}x$	Correct $\log_{10}x$
1.12		0.0492
1.16		0.0645
1.18		0.0719
1.21		0.0828
1.23		0.0899
1.24		0.0934
1.26		0.1004
1.28		0.1072
1.29		0.1106
1.31		0.1173
1.32		0.1206
1.33		0.1239
1.34		0.1271

3. Draw an accurate graph of the common logarithm function $y = \log_{10}x$ on a large sheet of graph paper. Use Table 9-1 as the table of values for drawing the graph. Compare your graph with the graph of $y = \log_{10}x$ in Fig. 9-1i.

4. Use sheets of graph paper similar to those in Fig. 9-1g and 9-1h to extend the graph of $y = 1/x$ both to the right and to the left. The class might undertake a cooperative project of drawing the graph from $x = 0.1$ to $x = 10$. This graph can be used to make a table of logarithms for all numbers from $x = 0.1$ to $x = 10$. Observe that the logarithms obtained are natural logarithms and not common logarithms as given in Table 9-1.

5. Use sheets of graph paper similar to those in Fig. 9-1g and 9-1h to extend the graph of $y = M/x$ in Fig. 9-1k both to the right and to the left. The class might undertake as a cooperative project the task of drawing the graph from $x = 0.65$ to $x = 10$. This graph can be used to make a table of common logarithms for all numbers from $x = 0.65$ to $x = 10$. Compare the values of $\log_{10}x$ obtained with the values given in Table 9-1.

6. If $\ln x$ is the natural logarithm of x, then $M \ln x = \log_{10}x$. Show that this relation can be used to compute a table of common logarithms from a table of natural logarithms.

7. Determine k in the equation of the hyperbola $y = \frac{k}{x}$ so that $\log\ 2 = 1$. We call this log function $\log_2 x$. Find the value of $\log_2 1$, $\log_2 3$, $\log_2 4$, $\log_2 8$, $\log_2 \frac{1}{2}$, and $\log_2 \frac{1}{4}$.

9-2. An Important Formula For Log x. The purpose of this section is to prove a theorem which states an important property of log x.

Theorem 9-2. If $y = \log x$ is the logarithm function derived from the hyperbola $y = k/x$, and if a and b are any two positive numbers, then

$$\log ab = \log a + \log b. \tag{9-2a}$$

Before we undertake to prove Theorem 9-2 let us verify Equation 9-2a in a number of special cases. Table 9-2a gives the values of $\log_{10}ab$ and $(\log_{10}a + \log_{10}b)$ for a number of different values of a and b. In three cases the two numbers differ by one in the fourth decimal place. A small difference of this size is to be expected occasionally since the logarithms in our table are correct to only four decimal places.

Table 9-2a. Comparison of $\log_{10}ab$ and $\log_{10}a + \log_{10}b$.

a	b	$\log_{10}ab$	$\log_{10}a + \log_{10}b$
1.50	2.00	0.4771	0.4771
2.50	3.00	0.8751	0.8750
1.50	3.00	0.6532	0.6532
2.00	2.50	0.6990	0.6989
3.00	4.00	1.0792	1.0792
6.50	2.00	1.1139	1.1139
3.00	5.00	1.1761	1.1761
4.00	4.00	1.2041	1.2042

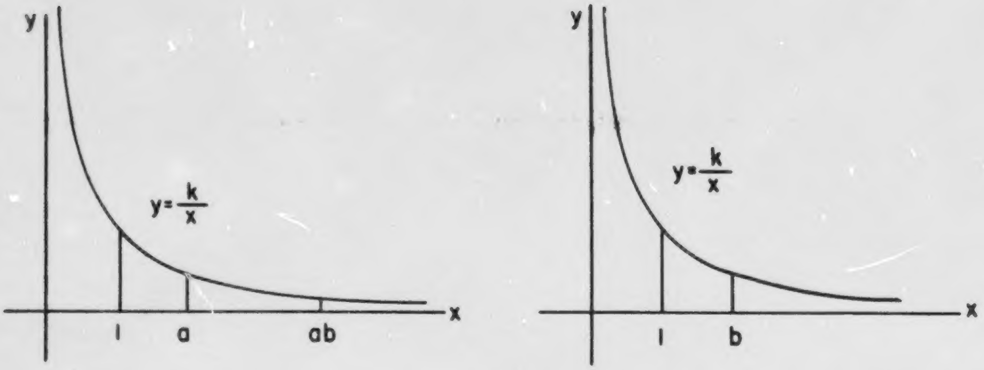

Figure 9-2a

Fig. 9-2a. The area under the hyperbola $y = k/x$ from $x = a$ to $x = ab$ is equal to the area under the hyperbola from $x = 1$ to $x = b$.

Consider the proof of Equation 9-2a. The graph on the left in Fig. 9-2a shows that the area under the hyperbola from $x = 1$ to $x = ab$ is $\log ab$, and that this area is equal to the area from $x = 1$ to $x = a$ plus the area from $x = a$ to $x = ab$. Since the area from $x = 1$ to $x = a$ is $\log a$ by definition, the proof of Equation 9-2a will be complete if we can show that the area under the hyperbola from $x = a$ to $x = ab$ is the same as the area from $x = 1$ to $x = b$. This fact will be proved in a special case; the proof in the general case can be given in the same way.

The proof will be given for $a = 2$ and $b = 3$. In this case we are asked to prove that the area under the hyperbola $y = k/x$ from $x = 2$ to $x = 6$ is equal to the area from $x = 1$ to $x = 3$. Approximate the latter area by four rectangles as shown in

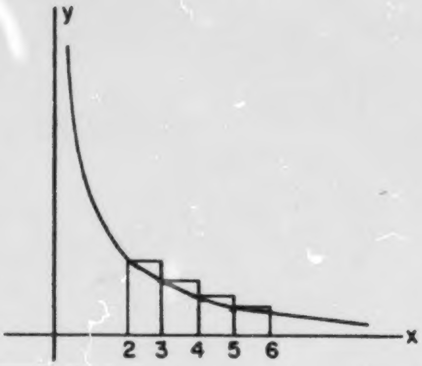

Figure 9-2b. Approximate area under the hyperbola $y = \frac{k}{x}$ from $x = 2$ to $x = 6$.

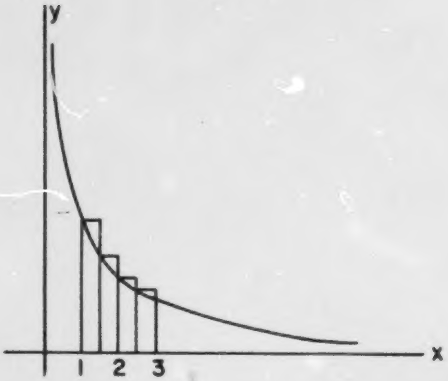

Figure 9-2c. Approximate area under the hyperbola $y = \frac{k}{x}$ from $x = 1$ to $x = 3$.

Fig. 9-2c, and approximate the former area by four corresponding rectangles as shown in Fig. 9-2b. The altitude of each rectangle can be found by calculating y from $y = k/x$ for the appropriate value of x. The calculations are shown in Table 9-2b. Observe that the area of each rectangle in Fig. 9-2b is exactly equal

Table 9-2b. Computation of the Areas in Figs. 9-2b and 9-2c.

Fig. 9-2b				Fig. 9-2c			
Rectangle	Length of Base	Altitude	Area	Rectangle	Length of Base	Altitude	Area
1	1	$\frac{k}{2}$	$\frac{k}{2}$	1	0.5	$\frac{k}{1.0}$	$\frac{k}{2}$
2	1	$\frac{k}{3}$	$\frac{k}{3}$	2	0.5	$\frac{k}{1.5}$	$\frac{k}{3}$
3	1	$\frac{k}{4}$	$\frac{k}{4}$	3	0.5	$\frac{k}{2.0}$	$\frac{k}{4}$
4	1	$\frac{k}{5}$	$\frac{k}{5}$	4	0.5	$\frac{k}{2.5}$	$\frac{k}{5}$

to the area of the corresponding rectangle in Fig. 9-2c. Thus the sum of the areas of the rectangles in Fig. 9-2b is equal to the sum of the areas in Fig. 9-2c. The same result will be found regardless of the number of rectangles used to approximate the areas. If a large number of rectangles is used, the sum of their areas is very close to the area under the curve. From these considerations it follows that the area under the hyperbola $y = k/x$ from $x = 2$ to $x = 6$ is equal to the area from $x = 1$ to $x = 3$.

A proof of Equation 9-2a for the general case can be given in exactly the same way.

Equation 9-2a has many applications. For example, Table 9-1 does not give $\log_{10}28$, but it does give $\log_{10}4$ and $\log_{10}7$. Therefore, by Equation 9-2a,

$$\begin{aligned}\log_{10}28 &= \log_{10}4 + \log_{10}7\\ &\approx 0.6021 + 0.8451\\ &\approx 1.4472.\end{aligned}$$

Observe also that $11 = (\sqrt{11})(\sqrt{11})$. Therefore,

$$\log_{10}11 = \log_{10}\sqrt{11} + \log_{10}\sqrt{11}$$

so that

$$\begin{aligned}\log_{10}\sqrt{11} &= \frac{1}{2}\log_{10}11\\ &\approx \frac{1}{2}(1.0414)\\ &\approx 0.5207.\end{aligned}$$

Exercises 9-2.

1. Verify Equation 9-2a in a number of special cases by completing the following table. Use the common logarithms given in Table 9-1.

a	b	ab	$\log_{10}ab$	$\log_{10}a + \log_{10}b$
3.00	3.00			
3.00	2.00			
4.00	2.50			
5.00	4.00			
5.00	7.00			
3.00	6.00			
6.00	5.00			
5.00	8.00			
5.00	10.00			
4.00	3.50			
5.00	9.00			

2. Use Equation 9-2a and Table 9-1 to calculate the values of the following logarithms:

(a) $\log_{10}21$
(b) $\log_{10}24$
(c) $\log_{10}22$
(d) $\log_{10}26$
(e) $\log_{10}27$
(f) $\log_{10}28$
(g) $\log_{10}32$
(h) $\log_{10}33$
(i) $\log_{10}34$
(j) $\log_{10}36$
(k) $\log_{10}38$
(ℓ) $\log_{10}42$
(m) $\log_{10}44$
(n) $\log_{10}48$
(o) $\log_{10}49$
(p) $\log_{10}51$
(q) $\log_{10}54$
(r) $\log_{10}56$
(s) $\log_{10}57$
(t) $\log_{10}63$
(u) $\log_{10}125$
(v) $\log_{10}144$
(w) $\log_{10}250$
(x) $\log_{10}1000$.

3. Prove that $\log a^2 = 2 \log a$. Use this fact to compute the following logarithms:

(a) $\log_{10}\sqrt{2}$
(b) $\log_{10}\sqrt{3}$
(c) $\log_{10}\sqrt{5}$
(d) $\log_{10}\sqrt{7}$
(e) $\log_{10}\sqrt{10}$
(f) $\log_{10}2.25$
(g) $\log_{10}6.25$
(h) $\log_{10}64$
(i) $\log_{10}81$
(j) $\log_{10}169$
(k) $\log_{10}256$
(ℓ) $\log_{10}441$
(m) $\log_{10}196$
(n) $\log_{10}289$
(o) $\log_{10}576$

4. Prove that $\log abc = \log a + \log b + \log c$ and thus that

$$\log a^2b = 2 \log a + \log b \quad \text{and}$$
$$\log a^3 = 3 \log a.$$

Use these facts to compute the following logarithms:

(a) $\log_{10}42$
(b) $\log_{10}1001$
(c) $\log_{10}255$
(d) $\log_{10}26.25$
(e) $\log_{10}(3.5)^2{\times}7$
(f) $\log_{10}147$
(g) $\log_{10}126.75$
(h) $\log_{10}343$
(i) $\log_{10}1728$
(j) $\log_{10}\sqrt[3]{5}$
(k) $\log_{10}\sqrt[3]{10}$
(ℓ) $\log_{10}\sqrt[3]{9.5}$
(m) $\log_{10}\sqrt[3]{20}$
(n) $\log_{10}\sqrt[3]{1000}$
(o) $\log_{10}\sqrt[3]{110.25}$

5. Use the definition of $\log x$ as an area to show that

$$\frac{k(x-1)}{x} < \log x < k(x-1) \quad \text{where} \quad k > 0 \quad \text{and} \quad x > 1.$$

Is this inequality true when $0 < x < 1$?

9-3. Properties Of log x.

Corresponding to the hyperbola $y = k/x$ there is a logarithm function $y = \log x$. This function was defined in Section 9-1. According to Definition 9-1, $\log x$ for each $k > 0$ has the following properties:

9-3a
$$\log 1 = 0,$$
$$\log x > 0, \quad x > 1,$$
$$\log x < 0, \ 0 < x < 1.$$

Furthermore, it was shown in Section 9-2 that

9-3b
$$\log(x_1 \cdot x_2) = \log x_1 + \log x_2.$$

In this section some additional properties of the logarithm function will be established.

In Equation 9-3b let x_1 be x, and let x_2 be $\frac{1}{x}$. Then

$$\log x(\tfrac{1}{x}) = \log x + \log \tfrac{1}{x}.$$

But since $x(\frac{1}{x}) = 1$, and $\log 1 = 0$ by Equation 9-3a, the last equation becomes

$$0 = \log x + \log \tfrac{1}{x}.$$

Thus it follows that

9-3c
$$\log \tfrac{1}{x} = - \log x.$$

Next, consider the logarithm of $\frac{x_1}{x_2}$. This quotient can be thought of as a product. Thus,

$$\frac{x_1}{x_2} = x_1(\tfrac{1}{x_2}).$$

Then by Equations 9-3b and 9-3c,

$$\log\frac{x_1}{x_2} = \log x_1 + \log \frac{1}{x_2},$$

or

9-3d
$$\log\frac{x_1}{x_2} = \log x_1 - \log x_2.$$

It will be shown next that if n is any positive integer, then,

9-3e
$$\log x^n = n \log x,$$
$$\log \frac{1}{x^n} = - n \log x.$$

The first statement in Equation 9-3e follows from repeated application of Equation 9-3b, for

$$\begin{aligned} \log x^2 &= \log(x \cdot x) \\ &= \log x + \log x \\ &= 2 \log x; \\ \log x^3 &= \log(x^2 \cdot x) \\ &= \log x^2 + \log x \\ &= 2 \log x + \log x \\ &= 3 \log x. \end{aligned}$$

The first statement in Equation 9-3e can be established by continuing in this fashion. The second statement follows from the first statement and from Equation 9-3c, for

$$\begin{aligned} \log \frac{1}{x^n} &= - \log x^n \\ &= - n \log x. \end{aligned}$$

The symbol $\sqrt[q]{x}$, where $x > 0$ and q is a positive integar means a positive number whose q^{th} power is x. Thus $(\sqrt[q]{x})^q = x$. For example, $\sqrt[3]{8} = 2$, $\sqrt[4]{16} = 2$, $\sqrt[6]{64} = 2$, $\sqrt[3]{125} = 5$. It will now be shown that if p and q are any positive integers, then

9-3f
$$\log(\sqrt[q]{x})^p = \frac{p}{q} \log x.$$

It will be shown first that $\log \sqrt[q]{x} = (\frac{1}{q}) \log x$. By the first statement in Equation 9-3e,

$$\log(\sqrt[q]{x})^q = q \log \sqrt[q]{x},$$

or

$$\log x = q \log \sqrt[q]{x}.$$

By solving this equation for $\log \sqrt[q]{x}$, we obtain

$$\log \sqrt[q]{x} = \frac{1}{q} \log x.$$

From this result and the first statement in Equation 9-3e it follows that

$$\log (\sqrt[q]{x})^p = \frac{p}{q} \log x,$$

and the proof of Equation 9-3f is complete.

The next property of $\log x$ to be established is the following:

9-3g If $x_1 < x_2$, then $\log x_1 < \log x_2$.

This property follows from the definition of $\log x$ (Definition 9-1); for, if $x_1 < x_2$, the area under the hyperbola $y = \frac{k}{x}$ from 1 to x_1 is less than the area under the curve from 1 to x_2.

A similar argument establishes

9-3g' If $\log x_1 < \log x_2$, then $x_1 < x_2$.

Note that statements 9-3g and 9-3g' can be expressed as one statement as follows:

$$x_1 < x_2 \text{ if and only if } \log x_1 < \log x_2$$

The next property of $\log x$ to be established is the following:

9-3h If $\log x_1 = \log x_2$, then $x_1 = x_2$.

There are only three possibilities: either $x_1 < x_2$, $x_1 > x_2$, or $x_1 = x_2$. But the first two are impossible, since $x_1 < x_2$ implies $\log x_1 < \log x_2$, and $x_1 > x_2$ implies $\log x_1 > \log x_2$ by Equation 9-3g. Thus, $x_1 < x_2$ and $x_1 > x_2$ must be rejected since both lead to contradictions. Therefore, $x_1 = x_2$, and the proof is complete.

Again, a similar argument establishes

9-3h' If $x_1 = x_2$, then $\log x_1 = \log x_2$

The final property of $\log x$ which is desired is the following:

9-3i The graph of $y = \log x$ is a continuous curve.

This follows from the fact that the graph has no breaks or jumps in it. An important consequence of this property is the following: If $x_1 < x_2$ and c is any number such that $\log x_1 < c < \log x_2$, then there is a number x_0 such that $x_1 < x_0 < x_2$ and $\log x_0 = c$.

The following is a summary of the properties of $\log x$:

9-3a $\log 1 = 0,$

$\log x > 0 \qquad x > 1,$

$\log x < 0 \qquad 0 < x < 1.$

9-3b If x_1 and x_2 are any two positive numbers, then

$$\log(x_1 \cdot x_2) = \log x_1 + \log x_2.$$

9-3c $\log \frac{1}{x} = - \log x.$

9-3d If x_1 and x_2 are any two positive numbers, then

$$\log \frac{x_1}{x_2} = \log x_1 - \log x_2.$$

9-3e If n is any positive integer, then

$$\log x^n = n \log x,$$

$$\log \frac{1}{x^n} = -n \log x.$$

9-3f If p and q are any two positive integers, then

$$\log (\sqrt[q]{x})^p = \frac{p}{q} \log x.$$

9-3g If $x_1 < x_2$, then $\log x_1 < \log x_2$.

9-3g' If $\log x_1 < \log x_2$, then $x_1 < x_2$

$$\left[x_1 < x_2 \text{ if and only if } \log x_1 < \log x_2\right]$$

9-3h If $\log x_1 = \log x_2$, then $x_1 = x_2$.

9-3h' If $x_1 = x_2$, then $\log x_1 = \log x_2$

$$\left[x_1 = x_2 \text{ if and only if } \log x_1 = \log x_2\right]$$

9-3i The graph of $y = \log x$ is a continuous curve.

Some applications of these properties of the function $\log x$ will be illustrated by examples:

Example 9-3a. Use Table 9-1 to find the following logarithms:

(a) $\log_{10} \frac{11^2}{17}$ (b) $\log_{10}(\sqrt[3]{5})^5$

Solution: (a) $\log_{10} \frac{11^2}{17} = \log_{10} 11^2 - \log_{10} 17$ (9-3d)

$\log_{10} 11^2 = 2 \log_{10} 11$ (9-3e)

$$\therefore \log_{10} \frac{11^2}{17} = 2 \log_{10} 11 - \log_{10} 17$$

$$\approx 2(1.0414) - 1.2304$$

$$\approx 0.8524$$

(b) According to (9-3f),

$$\log_{10}(\sqrt[3]{5})^5 = \frac{5}{3} \log_{10} 5$$

$$\approx \frac{5}{3}(0.6990)$$

$$\approx 1.1650$$

Example 9-3b. If $N = \frac{a^3 b}{\sqrt{c}}$, express $\log N$ in terms of the logarithms of a, b, and c.

Solution: $\log \frac{a^3 b}{\sqrt{c}} = \log a^3 b - \log \sqrt{c}$

$$= \log a^3 + \log b - \log \sqrt{c}$$

$$= 3 \log a + \log b - \frac{1}{2} \log c$$

Example 9-3c. Solve for x: $\frac{1}{3} \log x + \log 3 = \log 5$

Solution: $\log x + 3 \log 3 = 3 \log 5$

$$\log x = 3 \log 5 - 3 \log 3$$

$$\log x = \log 5^3 - \log 3^3$$

$$\log x = \log \frac{5^3}{3^3}$$

$$x = \frac{5^3}{3^3}$$

$$x = \frac{125}{27}$$

Example 9-3d. Solve for x: $\log(x-3) + \log x = \log 28$

Solution: Note that $\log a$ is not defined when $a \leq 0$. This means that we must have $(x - 3) > 0$ for our equation to be meaningful. We can write $\log[(x-3)x] = \log 28$. Since $x_1 = x_2$ if $\log x_1 = \log x_2$, we have $(x-3)x = 28$ or $x^2 - 3x - 28 = 0$. The roots of this quadratic are found to be 7 and -4. We observe that 7 satisfies the original equation and that -4 must be rejected for the reason we have indicated.

Exercises 9-3.

1. Use Table 9-1 and the properties of $\log x$ stated in this section to find the following logarithms:

   (a) $\log_{10} \frac{5}{7}$

   (b) $\log_{10}(4 \times 7.5)$

   (c) $\log_{10}(\frac{1}{4} \times 17)$

   (d) $\log_{10} \frac{1.50 \times 3.50}{2.50}$

   (e) $\log_{10} (13)^6$

   (f) $\log_{10} \left(\frac{5}{\sqrt{13}}\right)$

   (g) $\log_{10} \frac{(2.5)^3(3.5)^5}{\sqrt{7}}$

   (h) $\log_{10}(\sqrt[3]{12})^4$

   (i) $\log_{10}\sqrt{\frac{(3.5)^2 \times (5.5)^3}{45}}$

   (j) $\log_{10} \frac{1}{\sqrt[3]{7}}$

   (k) $\log_{10} (3^2 + 4^2)$

   (ℓ) $\log_{10} \frac{1}{3^3 \cdot \sqrt[4]{7}}$

   (m) $\log_{10} \frac{1}{2^3 + 3^3}$

2. Find the value of 2×7 by the use of logarithms.

Solution: By Equation 9-3b and Table 9-1,

$$\begin{aligned}\log_{10}(2 \times 7) &= \log_{10}2 + \log_{10}7 \\ &\approx 0.3010 + 0.8451 \\ &\approx 1.1461.\end{aligned}$$

Table 9-1 shows that

$$\log_{10}14 \approx 1.1461.$$

It follows from Equation 9-3h that $2 \times 7 = 14$.

3. Find the value of $\frac{2.50 \times 18.00}{4.50}$ by using logarithms.

4. If $N = \frac{15 \times 8}{3}$, find N by means of logarithms.

5. Express the logarithms of each of the following expressions in terms of the logarithms of the letters involved as in Example 2:

(a) PQR

(b) $\frac{P(\sqrt[3]{Q})^2}{R}$

(c) $\frac{Q}{P^2R^3}$

(d) $\frac{\sqrt{PQ}}{R}$

(e) $\sqrt{\frac{PQ}{R}}$

(f) $\sqrt[3]{\frac{P^2Q}{R^5}}$

(g) $\frac{1}{P\sqrt{Q}}$

(h) $\frac{1}{2}\sqrt{\frac{Q}{R^3}}$

6. Solve each of the following logarithmic equations for x:

   (a) $\log_{10} x = 3 \log_{10} 7$

   (b) $\log_{10} x + \log_{10} 13 = \log_{10} 182$

   (c) $2 \log_{10} x - \log_{10} 7 = \log_{10} 112$

   (d) $\log_{10} (x-2) + \log_{10} 5 = 2$

   (e) $\log_{10} x + \log_{10} (x + 3) = 1$

   (f) $\frac{1}{2} \log_{10} x = - \log_{10} 64$

   (g) $\log_{10} (x-2) + \log_{10} (x + 3) = \log_{10} 14$

7. Write without "log":

   (a) $\log_{10} V = \log_{10} 4 + \log_{10} \pi + 3 \log_{10} r - \log_{10} 3$

   (b) $\log_{10} P = \frac{1}{2} \log_{10} t + \frac{1}{2} \log_{10} g$

   (c) $\log_{10} S = \frac{1}{2} \left[\log_{10} s + \log_{10} (s-a) + \log_{10} (s-b) + \log_{10} (s-c)\right]$

8. Show by logarithms that, if $a > 0$, and p, q, and n are natural numbers, then

   (a) $\sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

   (b) $\sqrt[nq]{a^n} = \sqrt[q]{a}$ Hint: Use Property 9-3f

9. Express as a single logarithm:

   (a) $\log_{10} x + \log_{10} y - \log_{10} z$

   (b) $\log_{10} (x + 3) - \log_{10} (x-2)$

   (c) $4 \log_{10} t - 3 \log_{10} s$

   (d) $\frac{1}{2} \log_{10} x - \frac{2}{3} \log_{10} y$

   (e) $\log_{10} 2 + \log_{10} x + 3(\log_{10} x - \log_{10} y)$

   (f) $- \log_{10} x + 4 \log_{10} (x-2) + \frac{1}{3} \log_{10} x^2$

*10. Suppose we denote the area under the curve $y = 3x^2$ in the first quadrant between the ordinates at 1 and x as "lug x" Are there any properties of log x which are also true for "lug x"? In particular is it true that lug ab = lug a + lug b?

9-4. The Graph of $y = \log x$.

Fig. 9-11 contains graphs of $y = \ln x$ and $y = \log_{10}x$. These graphs exhibit many of the characteristic features and important properties of all logarithm functions. This section will be devoted to a study of the properties of the graph of the general logarithm function $y = \log x$.

The first important property of the graph of $y = \log x$ is this: The ordinate y always increases as x increases. It was proved in 9-3g that, if $x_1 < x_2$, then $\log x_1 < \log x_2$. The fact that y always increases as x increases on the graph of $y = \log x$ is a consequence.

It follows from the definition of log x that $\log 1 = 0$ (see also Equation 9-3a). Thus the graph of $y = \log x$ crosses the x-axis at (1,0). The graph does not cross the x-axis at any other point because y always increases as x increases.

It has been explained already that the graph of $y = \log x$ is a continuous curve (see Property 9-31). The graphs of $y = \ln x$ and $y = \log_{10} x$ in Fig. 9-11 are continuous curves, and the graph of every logarithm function $y = \log x$ has this same property.

Another important property of the graph of $y = \log x$ is the following: As x increases without limit, y also increases without limit. By 9-3g, we know that if $x_1 < x_2$, then $\log x_1 < \log x_2$. Since $1 < 2$, $\log 1 < \log 2$, and $0 < \log 2$. Consider $\log 2^n$. Since $\log 2^n = n \log 2$, it follows that $\log 2^n$ increases without limit as n increases without limit. Thus, the point $(2^n, n \log 2)$ is on the graph of $y = \log x$, and the ordinate of this point is arbitrarily large if n is sufficiently large. Since y always increases as x increases, it follows that y increases without limit as x increases on the graph of $y = \log x$.

A closely related property is the following: As x decreases toward zero, y decreases without limit on the graph of $y = \log x$ Another way to state this property is the following: the graph of $y = \log x$ is asymptotic to the negative y-axis. It follows from Equation 9-3e that $\log \frac{1}{2^n} = - n \log 2$.

Thus, the point $(\frac{1}{2^n}, - n \log 2)$ is on the graph of $y = \log x$. As n increases without limit, the abscissa of this point decreases toward zero, and the ordinate decreases without limit. Since y always decreases as x decreases, the graph of $y = \log x$ is asymptotic to the negative y-axis as stated.

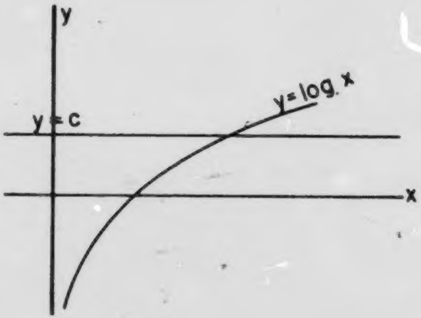

Figure 9-4a. The graph of $y = \log x$ crosses every line $y = c$ once and only once.

The final property of the graph of $y = \log x$ is the following: If c is any real number, then the graph of

$y = \log x$ crosses the line $y = c$ at one and only one point. This property is an important consequence of the fact that the graph of $y = \log x$ is continuous, and Fig. 9-4a gives a graphical proof of it. The figure shows the graphs of $y = \log x$ and $y = c$. If the graph of $y = \log x$ crosses the graph of $y = c$ once, then the curved line cannot cross the straight line a second time because the ordinate y on the graph of $y = \log x$ always increases as x increases. Thus, the proof will be complete if it can be shown that the graph of $y = \log x$ crosses the line at least once. It has been shown already in this section that there is a point on the graph of $y = \log x$ above the line $y = c$ and another point below this line. Since the graph of $y = \log x$ is continuous, the graph crosses the line in passing from the point below the line $y = c$ to the point above this line. The proof is complete.

Another statement of the property proved in the last paragraph is the following: If c is any real number, then there is exactly one positive real number x_0 such that $\log x_0 = c$.

The following is a summary of the properties of the graph of $y = \log x$ established in this section.

9-4a On the graph of $y = \log x$, the ordinate y always increases as the abscissa x increases.

9-4b The graph of $y = \log x$ crosses the x-axis at $x = 1$ and at no other point.

9-4c The graph of $y = \log x$ is a continuous curve.

9-4d As x increases without limit, y also increases without limit on the graph of $y = \log x$.

9-4e As x decreases toward zero, y decreases without limit on the graph of $y = \log x$.

9-4f If c is any real number, then the graph of $y = \log x$ crosses the line $y = c$ at one and only one point.

Exercises 9-4

1. Find the coordinates of a point P on the graph of $y = \log_{10}x$ which satisfies each of the following conditions:

   (a) The ordinate of P is greater than 100.

   (b) The ordinate of P is less that -5.

   (c) The ordinate of P is greater than 1 and less than 2.

   Hint for (a): Recall that $\log_{10}10 = 1$ and that

   $$\log_{10}10^n = n \log_{10}10.$$

2. Draw an accurate graph of $y = \log_{10}x$ on a large sheet of graph paper (see Exercise 9-1-3). Use this graph to find the approximate solutions of the following equations: (Note that graph must extend to at least $x = 100$).

   (a) $\log_{10}x = .5$

   (b) $\log_{10}x = .8$

   (c) $\log_{10}x = -1$

   (d) $\log_{10}x = 0$

   (e) $\log_{10}x = -2.5$

   (f) $\log_{10}x = 1.2$

   (g) $\log_{10}x = 2$

   (h) $\log_{10}x = \sqrt{2}$

   (i) $\log_{10}x = \frac{\sqrt{2}}{2}$

   (j) $\log_{10}x = \frac{\sqrt{3}}{3}$

3. We label the log function whose value at 10 is 1 with the symbol $\log_{10} x$. Similarly, the log function whose graph passes through the point $(t,1)$ is called $\log_t x$. Find the value of k associated with $\log_t x$. Show also that $\log_t t^n = n$, where n is any positive integer.

4. If the graph of $y = \log x$ passes through the point (t,s) where $t > 1$, show that $\log t^n = ns$ for any positive integer n. Also find the value of k such that the graph of $y = \log x$ passes through (t,s).

5. Show that there exists a number x such that $\log_{10} x = \pi$ Show also that this number is greater than 1000 and less than $1000 \sqrt{10}$.

6. Sketch a curve which has the property that it is symmetric to the graph of $y = \log x$ with respect to the line $y = x$. Suppose that the equation of this new curve is $y = E(x)$.

   (a) Re-state properties 9-4a, 4b, 4c, 4d as they apply to the graph of $y = E(x)$.

   (b) Which of the following are true?

      (1) The functions $\log x$ and $E(x)$ are so related that the domain of either function is the range of the other.

      (2) If $P(a,b)$ lies on either graph, then the point $Q(b,a)$ lies on the other.

9-5. Tables of Common Logarithms; Interpolation.

It was shown in Table 9-1 that the common logarithms of a few numbers are integers; for example, $\log_{10} 0.01 = -2$, $\log_{10} 1 = 0$, and $\log_{10} 10 = 1$. The common logarithms of some numbers are rational fractions; for example, $\log_{10} \sqrt{10} = \frac{1}{2}$ (see Equation 9-3f). The common logarithms of many numbers are irrational numbers; for example, the number $10^{\sqrt{2}}$ will be defined later, and it will be shown that $\log_{10} 10^{\sqrt{2}}$ is the irrational number $\sqrt{2}$. The usual tables of logarithms express approximate values of the logarithms of numbers in decimal form correct to four, five, or seven decimal places.

Four-place tables will be used in this section and the next.

Table 9-5a. Approximations to a Few Common Logarithms and Their Representation in Standard Form.

x	$\log_{10}x$	$\log_{10}x$ in Standard Form
0.00231	- 2.6364	- 3 + .3636
0.0231	- 1.6364	- 2 + .3636
0.231	- 0.6364	- 1 + .3636
2.31	0.3636	0 + .3636
23.1	1.3636	1 + .3636
231.0	2.3636	2 + .3636
2310.0	3.3636	3 + .3636
23100.0	4.3636	4 + .3636

It has been shown that the logarithms of numbers greater than 1 are positive, and that the logarithms of numbers less than 1 are negative. The second column of Table 9-5a gives the common logarithms of numbers listed in the first column. The third column shows the logarithms written in standard form. It will be observed that $\log_{10}x$, when written in standard form, is the sum of an integer (positive, negative or zero) and a non-negative decimal fraction less than 1. The integer is called the characteristic of the logarithm, and the decimal fraction is called the mantissa. Thus, the standard form for writing the common logarithm of a number a is

Definition 9-5a. $\log a = n + m$, where
n is a positive or negative integer or zero, and
$0 \leq m < 1$.

We illustrate the meaning of this definition with some examples:

Example 9-5a. Find the characteristic n and the mantissa m of $\log_{10} a$ for each of the following values:

(a) $\log_{10}a = .4829$

Solution: It is important to observe that the characteristic n can be zero as it is in this case. We can write:

$\log_{10}a = .4829 = 0.4829 = 0 + .4829$ where

$n = 0$ and $m = .4829$. Note $0 \leq m < 1$.

(b) $\log_{10}a = 3.3122 + 1.5040$

Solution: Clearly $\log_{10}a = 4.8162 = 4 + .8162$, therefore, $n = 4$ and $m = .8162$. Again, $0 \leq m < 1$.

(c) $\log_{10}a = -2.4163$

Solution: If we write $\log_{10}a = -2 + (-.4163)$, we observe that the decimal fraction is negative and therefore cannot be regarded as a mantissa which, by definition, is a non-negative number less than one. In this case $\log a$ is larger than -3 and less than -2. This means that $\log_{10}a$ can be expressed as -3 plus some positive number less than one. This positive number is our mantissa m.

$\log_{10}a = -3 + m$ or $-2.4163 = -3 + m$. $m = .5837$. This gives $\log_{10}a = -3 + .5837$. We see that $n = -3$ and that $0 \leq m < 1$. Note that we could have obtained this result more quickly by adding and subtracting 3:

$$\log_{10}a = -2.4163 = -2.4163 + 3 - 3 = .5837 - 3 = -3 + .5837.$$

Example 9-5b. Find the characteristic n and the mantissa m for

$\log_{10}a$ if $5 \log_{10}a = 2 \log_{10}x - 3 \log_{10}y$, where

$\log_{10}x = 0.1962$, and $\log_{10}y = 0.7343 - 2$.

Solution:

$$\begin{aligned}5 \log_{10} a &= 2 \times (0.1962) - 3(0.7343 - 2)\\ &= 0.3924 - 3(-1.2657)\\ &= 0.3924 + 3.7971\\ &= 4.1895\\ \log_{10} a &= 0.8379. \quad n = 0 \text{ and } m = .8379.\\ & \qquad 0 \le m < 1\end{aligned}$$

Exercises 9-5a

Find the characteristic and the mantissa for $\log_{10} a$ in Exercises 1-12:

1. $\log_{10} a = 3.8383$
2. $\log_{10} a = .5332$
3. $\log_{10} a = -.4431$ Hint: $\log_{10} a = -1 + ?$
4. $\log_{10} a = -2.2136$
5. $\log_{10} a = -5$
6. $\log_{10} a = -1.3166$
7. $\log_{10} a = .2727 - 3.8122$
8. $\log_{10} a = .4177 + 1.7832 - 5$
9. $\log_{10} a = -.0926$
10. $3 \log_{10} a = -4$
11. $2.6183 + \log_{10} a = 1.2336$
12. $\log_{10} a = \frac{1}{2}\left[3 \log_{10} x + \log_{10} y - \frac{1}{5} \log_{10} z\right]$ where

    $\log_{10} x = 0.3163$, $\log_{10} y = -.8887$, $\log_{10} z = -7.4175$

13. Let n represent the characteristic and m the mantissa for $\log_{10}a$, $a > 0$. Is the following statement true?: If $\log_{10}a = 0$, then $m = 0$ and $n = 0$.

14. Are these statements true?
    (a) If $\log_{10}a$ and $\log_{10}b$ have the same mantissas, then they differ by an integer.
    (b) If $\log_{10}a$ and $\log_{10}b$ differ by an integer, then their mantissas are equal.

Let us now consider two positive numbers whose decimal representations differ only in the position of the decimal point. We see that 73.18 and $.07318$ are a pair of numbers of this type. In this case, we note that $73.18 = .07318 \times 10^3$.

The sample of logarithms given in Table 9-5a suggests that the common logarithms of any two numbers whose decimal representations differ only in the positions of the decimal points have the same mantissas. This fact will be proved in the following theorem:

Theorem 9-5a. If a and b are any two positive numbers whose decimal representations differ only in the positions of the decimal points, then $\log_{10}a$ and $\log_{10}b$ have the same mantissas.

The proof employs the properties of logarithms established in Section 9-3. For convenience, assume that $a > b$; a similar proof can be given if $b > a$. Then there exists a positive integer n such that

$$a = 10^n b.$$

Recall that $\log (x_1 \cdot x_2) = \log x_1 + \log x_2$, $\log x^n = n \log x$, and $\log_{10}10 = 1$.

Then,

$$\begin{aligned}\log_{10}a &= \log_{10}(10^n b)\\ &= \log_{10}10^n + \log_{10}b\\ &= n\log_{10}10 + \log_{10}b\\ &= n + \log_{10}b.\end{aligned}$$

Thus, $\log_{10}a$ is obtained by adding the integer n to $\log_{10}b$, and the mantissas of $\log_{10}a$ and $\log_{10}b$ are the same. The proof is complete.

It follows from Theorem 9-5a that the logarithms of all numbers can be obtained from a table which gives the logarithms of numbers from 1 to 10. Common logarithms are preferred to natural logarithms for ordinary computation because of Theorem 9-5a. The mantissas of common logarithms are obtained from a table, and characteristics are obtained by inspection as indicated in the next Theorem (9-5b).

Before we consider this theorem, let us recall the meanings that have been assigned to such expressions as 10^0, and 10^{-3} and 10^{-n} where n is a positive integer. We have long known that $10^3 \times 10^2 = (10 \times 10 \times 10)(10 \times 10) = 10^5$ and, more generally, that

(1) $10^m \times 10^n = 10^{m+n}$, where m and n are positive integers called exponents when used in this way.

Zero and negative integral exponents were defined so that this law (1) remains true. Suppose $n = 3$ and $m = 0$. We have $10^3 \times 10^0 = 10^{3+0} = 10^3$. Evidently 10^0 must be assigned the value 1 in order for the statement to be true. We can write

$$10^0 = 1.$$

Suppose next that $m = 3$ and $n = -3$. According to our rule, we have

$$10^3 \times 10^{-3} = 10^{3+(-3)} = 10^0 = 1.$$

But $10^3 \times \frac{1}{10^3} = 1$. Therefore, 10^{-3} must be interpreted as $\frac{1}{10^3}$ in order for (i) to remain true for negative integral exponents. We have then $10^{-3} = \frac{1}{10^3}$ and, in general,

$$10^{-n} = \frac{1}{10^n} \quad \text{where } n \text{ is a positive}$$

integer. In fact, for any real number $x \neq 0$, we have

$$x^{-n} = \frac{1}{x^n} \quad \text{where } n \text{ is a positive integer.}$$

We note that Equations 9-3e can now be written as a single equation:

9-5b $\log x^n = n \log x$ for any integral value of n provided $x > 0$.

The use of zero and negative integers as exponents will be illustrated by examples:

Example 9-5c. Express in decimal form:

(a) 10^{-5}

(b) $10^0 \times 10^{-3}$

(c) 416.2×10^{-5}

Solutions:

(a) $10^{-5} = \frac{1}{10^5} = \frac{1}{100,000} = 0.00001$

(b) $10^0 \times 10^{-3} = 1 \times \frac{1}{1000} = 0.001$

(c) $416.2 \times 10^{-5} = 416.2 \times \frac{1}{100,000} = 0.004162$

Example 9-5d. Supply the appropriate exponent:

(a) $0.00001 = 10^x$

(b) $10^3 \times 10^4 \times 10^{-7} = 10^y$

(c) $0.0512 = 5.12 \times 10^z$

Solutions: The definitions indicate that the answers are

$$x = -5, \quad y = 0 \quad \text{and} \quad z = -2.$$

This brief discussion of integral exponents is sufficient for our present purpose. A more complete discussion of exponents is found in Section 8 of this Chapter along with a number of practice exercises.

We are now in a position to establish

Theorem 9-5b. If N is a positive number expressed in the form $10^n \times k$, where n is an integer and $1 \leq k < 10$, then n is the characteristic of $\log_{10}N$.

Proof:

1. $N = 10^n \times k$ Hypothesis
2. $\log_{10}N = \log_{10}(10^n \times k)$ (9-3h)
3. $\log_{10}N = \log_{10} 10^n + \log_{10}k$ (9-3b)
4. $\log_{10}N = n + \log_{10}k$ (9-3e) and $\log_{10}10 = 1$ by definition
5. $1 \leq k < 10$ Hypothesis
6. $\log_{10} 1 \leq \log_{10}k < \log_{10}10$ (9-3g)
7. $0 \leq \log_{10} k < 1$ $\log_{10}1 = 0$ and $\log_{10}10 = 1$
8. n is the characteristic of $\log_{10}N$ by definition (9-5a)

Q.E.D.

Examples will show how this Theorem can be applied as well as the significance of our preliminary note on zero as an exponent.

Example 9-5e. Find the characteristic n of $\log_{10}N$ for each of the following values of N:

(a) $N = 4513$

(b) $N = 0.00847$

(c) $N = 7.418$

Solutions:

(a) $N = 4.513 \times 10^3$, therefore $n = 3$ by Theorem 9-5b.

(b) $N = 8.47 \times 10^{-3}$; $\therefore n = -3$.

(c) $N = 7.418 \times 1 = 7.418 \times 10^0$; $\therefore n = 0$.

Example 9-5f. If we know that the characteristic of $\log_{10}N$ is 2, and that the sequence of digits in N is 4821, locate the decimal point in N.

Solution: In our formula $N = 10^n \times k$, we have $k = 4.821$ and $n = 2$. $\therefore N = 4.821 \times 10^2 = 482.1$

Exercises 9-5b (Oral)

1. Give the characteristic for $\log_{10}N$ for each of the following values of N:

(a) 43.16

(b) $763{,}900$

(c) 7.732

(d) 0.7732

(e) 0.000085

(f) $10^{-8} \times 6.32$

(g) 471.5×10^4

(h) 0.0063×10^3

(i) 6315×10^{-7}

(j) $10^5 \times 10^3 \times 10^{-2}$

2. In each of the following cases we are given the sequence of digits in N and the characteristic of $\log_{10}N$. Locate the decimal point (find N) in each case.

	Sequence of Digits in N	Characteristic of $\log_{10}N$
(a)	77113	5
(b)	63192	0
(c)	2083	-3
(d)	5331	-7
(e)	29003	2

3. A number in decimal form is said to have its decimal point in standard position if the decimal point is located just to the right of the first non-zero digit. Use this idea along with Theorem 9-5b to obtain a rule for finding the characteristic of the logarithm of any number which has been expressed in decimal form.

4. Apply the rule you obtained in Exercise 3 to find the characteristic of log N when N is given as follows:

(a) 417800

(b) 0.0031

(c) 731×10^{-5}

(d) 0.001×0.0002

(e) between 0.001 and 0.009

(f) $4 \times 10^3 + 273$

(g) $2.16 \times 10^3 \times 3.19 \times 10^{-3}$

It is now possible to explain how the logarithm of a number is obtained from a table. Table 9-5b shows a small portion of a standard four-place table of common logarithms. The first two digits of the number are given in the column on the left which is headed N; the third digit appears at the tops of the columns on the right. A complete four-place table appears at the end of this Section (Table 9-5d). These tables give only mantissas, and all decimal points are omitted. Characteristics are obtained by applying Theorem 9-5b. Table 9-5c gives a number of logarithms that have been obtained in the manner indicated.

Table 9-5b. Sample Entries from a Four-Place Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254

If the logarithm of a number is known, the digits of the number can be found by looking in a table of logarithms. If the given logarithm is a common logarithm, look for the mantissa in the body of a table of common logarithms and read the digits of the number at the left margin and at the top of the column in which the mantissa is found. The characteristic indicates where the decimal point should be placed. Table 9-5c can also be interpreted as giving examples of how to find the number a when $\log_{10}a$ is given.

Table 9-5c. Some Common Logarithms Obtained from Table 9-5b.

a	$\log_{10}a$
6180.	3 + .7910
6160.	3 + .7896
62.1	1 + .7931
6.15	0 + .7889
6.18	0 + .7910
0.619	-1 + .7917
0.0619	-2 + .7917
0.00619	-3 + .7917
619.	2 + .7917
6190.	3 + .7917
61900.	4 + .7917
617000.	5 + .7903
6.21	0 + .7931

The discussion of tables of logarithms will be complete as soon as interpolation has been described. Consider the problem of finding $\log_{10}621.6$. Inspection of the tables shows readily that

$$\log_{10}621.0 \approx 2.7931,$$
$$\log_{10}622.0 \approx 2.7938,$$

but the digits 6216 do not occur in a standard four-place table. Since $\log x_1 < \log x_2$ if $x_1 < x_2$,

$$2.7931 < \log_{10}621.6 < 2.7938,$$

but further information is needed to find $\log_{10}621.6$. An examination of the graph of $y = \log_{10}x$ in Fig. 9-1i shows that short sections of the graph are almost straight. More precisely, let

P_1 and P_2 be two points on the graph of $y = \log_{10}x$ which lie close together; then the segment of a straight line that joins these two points lies very close to the graph of $y = \log_{10}x$. Thus, in order to find $\log_{10}621.6$, the graph of $y = \log_{10}x$ will be approximated by the straight line through the points (621.0, 2.7931) and (622.0, 2.7938).

Fig. 9-5a gives a schematic drawing which explains how the straight line is used to obtain an approximate value for $\log_{10}621.6$.

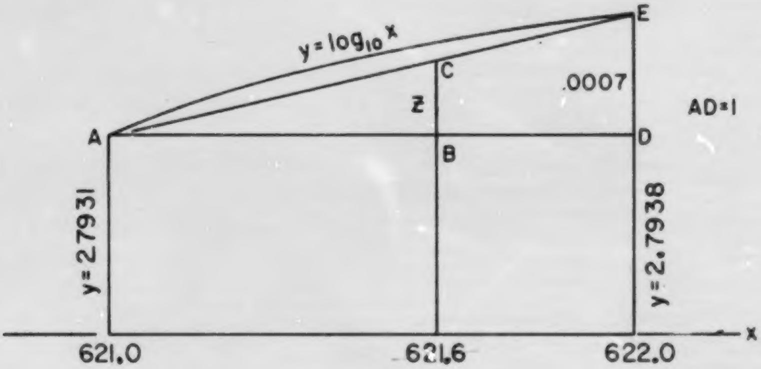

Fig. 9-5a. Explanation of Linear Interpolation

Observe from the figure that the logarithm increases by .0007 (the number 7 is usually called the tabular difference) when x increases by 1.0. The triangles ABC and ADE are similar. Therefore,

$$\frac{m(\overline{BC})}{m(\overline{AB})} = \frac{m(\overline{DE})}{m(\overline{AD})} \text{ or } \frac{z}{0.6} = \frac{.0007}{1.0}$$

and $z = .00042$. This number must be rounded off to .0004 since it is not possible to obtain five-place accuracy by interpolating in a four-place table. Finally, add .0004 to 2.7931 to obtain $\log_{10}621.6 \approx 2.7935$.

The process of finding the mantissa for the logarithm of a number whose digits occur between two entries in the table is called linear interpolation because a straight line is used to approximate the graph of $y = \log_{10}x$.

The problem of finding $\log_{10}621.6$ can also be solved very simply by finding the equation of the line through the two points A and E in Fig. 9-5a. The figure shows that A and E have the coordinates (621.0, 2.7931) and (622.0, 2.7938) respectively. The equation of the line through A and E is

$$y - 2.7931 = \frac{2.7938 - 2.7931}{622.0 - 621.0}(x - 621.0),$$

or

$$y = 2.7931 + .0007\ (x - 621.0).$$

The value of y when $x = 621.6$ is the approximate value of $\log_{10}621.6$. If $x = 621.6$, the last equation gives $y = 2.7935$. Hence, $\log_{10}621.6 \approx 2.7935$.

It is often necessary to interpolate in order to find a number when its logarithm is known. For example, consider the problem of finding x if

$$\log_{10}x = 1.7940.$$

Table 9-5b shows that

$$\log_{10}62.2 \approx 1.7938$$

$$\log_{10}62.3 \approx 1.7945,$$

but the mantissa 0.7940 does not occur in the table.

Fig. 9-5b gives a schematic diagram which indicates how the graph of $y = \log_{10}x$ can be approximated by a straight line in such a way as to give a solution to the problem. Similar triangles give the equation

$$\frac{z}{.0002} = \frac{0.1}{.0007}, \text{ or } z = \frac{2}{7}(0.1).$$

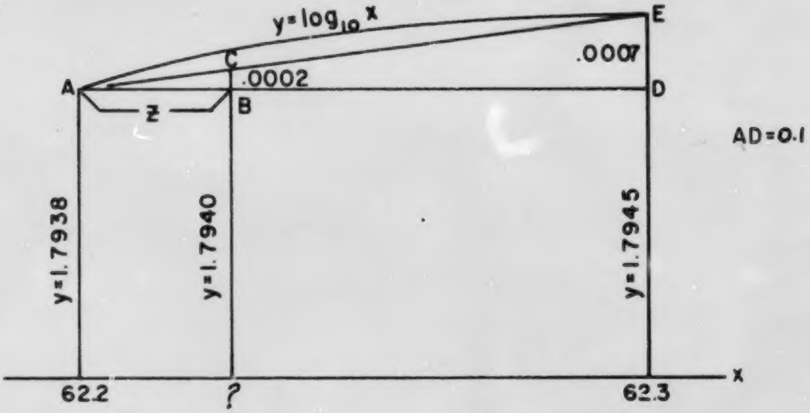

Fig. 9-5b. Explanation of linear interpolation for finding a number when its logarithm is given.

Thus, z is approximately .03, and the number whose common logarithm is 1.7940 is approximately 62.23.

The problem just explained can be solved also by finding the equation of the straight line AE in Fig. 9-5b. This line passes through the points whose coordinates are (62.2, 1.7938) and (62.3, 1.7945). The equation of this line is

$$y - 1.7938 = .007\ (x - 62.2).$$

If $y = 1.7940$ on this line, then $x = 62.23$. Thus, if $\log_{10}x = 1.7940$, then $x \approx 62.23$.

Table 9-5d. FOUR-PLACE TABLE OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Exercises 9-5c.

Use Table 9-5d with the following exercises.

1. Find the logarithm of each of the following numbers:

   (a) 342.0
   (b) 38.4
   (c) .735
   (d) .0945
   (e) 58900
   (f) 21.4
   (g) 349.0
   (h) .549
   (i) .00684
   (j) 734000
   (k) 9450
   (ℓ) 73.2
   (m) .000654
   (n) 7.68
   (o) 8.62

2. Find the logarithm of each of the following numbers. Interpolation is required.

   (a) 684.2
   (b) 9.484
   (c) .06254
   (d) .7328
   (e) 271.6
   (f) 1.647
   (g) 38.74
   (h) 495500.
   (i) .05879
   (j) .0006237
   (k) 788600000.
   (ℓ) 8.589

3. Find the numbers that have the following logarithms:

   (a) 2 + .4425
   (b) 2.4425
   (c) -2 + .8274
   (d) -2.7167
   (e) 4 + .6646
   (f) 0 + .3522
   (g) 1 + .2330
   (h) -3 + .6839
   (i) -3.2924
   (j) 3.7135

4. Find the numbers that have the following logarithms. Interpolation is required.

   (a) 2 + .4505
   (b) -1 + .9156
   (c) 4 + .1320
   (d) 5.3328
   (e) -2 + .4748
   (f) -2.4748
   (g) -2 + .7592
   (h) 1 + .8487
   (i) -1 + .6329
   (j) 3 + .4279

5. Draw an accurate graph of $y = \log_{10}x$ on a large sheet of graph paper. Use Table 9-5d as a table of values for plotting the graph.

9-6. Computation With Common Logarithms.

Computation with common logarithms rests on two simple facts:

(a) A number can be found (by using tables) if its logarithm is known.

(b) By using the properties of logarithms established in Section 9-3, it is frequently possible to find the logarithm of a complicated expression quite simply from the logarithms of the individual numbers in the expression.

The procedure is best explained by means of examples. Since all logarithms in this section are common logarithms, the subscript 10 has been omitted from the symbol $\log_{10}a$ in order to simplify writing.

Example 9-6a. Find the value of 27.43×71.64.

Solution:

Let a denote the value of the expression. Then by the properties of logarithms established in Section 9-3,

$$\begin{aligned}\log a &\approx \log (27.43 \times 71.64)\\ &\approx \log 27.43 + \log 71.64.\end{aligned}$$

In order to make the addition easy, the work may be arranged in tabular form as follows:

$$\begin{aligned}\log 27.43 &\approx 1.4383\\ \underline{\log 71.64} &\approx \underline{1.8551}\\ \log (27.43 \times 71.64) &\approx 3.2934\\ 27.43 \times 71.64 &\approx 1965.\end{aligned}$$

Example 9-6b. Find the value of $\frac{71.64}{25.64}$

Solution: By the properties of logarithms established in Section 9-3,

$$\log \frac{71.64}{25.64} = \log 71.64 - \log 25.64.$$

$$\begin{aligned} \log 71.64 &\approx 1.8551 \\ \underline{\log 25.64} &\approx \underline{1.4089} \\ \log \frac{71.64}{25.64} &\approx 0.4462 \\ \frac{71.64}{25.64} &\approx 2.794. \end{aligned}$$

Example 9-6c. Find the value of $\frac{27.43 \times (71.64)^2}{(25.64)^3}$

Solution: Let a denote the value of the expression. Then by the properties of logarithms established in Section 9-3,

$$\begin{aligned} \log a &= \log\,[27.43 \times (71.64)^2] - \log\,(25.64)^3 \\ &= \log 27.43 + 2 \log 71.64 - 3 \log 25.64 \end{aligned}$$

In order to make the additions and subtractions easy, the work should be arranged in tabular form as follows:

$$\begin{aligned} \log 27.43 &\approx 1.4383 \\ \underline{2 \log 71.64} &\approx \underline{3.7102} \\ \log[27.43 \times (71.64)^2] &\approx 5.1485 \\ \underline{3 \log 25.64} &\approx \underline{4.2267} \\ \log \frac{27.43\,(71.64)^2}{(25.64)^3} &\approx 0.9218 \\ \frac{27.43\,(71.64)^2}{(25.64)^3} &\approx 8.352 \end{aligned}$$

$$\boxed{\begin{aligned} \log 71.64 &\approx 1.8551 \\ 2 \log 71.64 &\approx 3.7102 \\ \log 25.64 &\approx 1.4089 \\ 3 \log 25.64 &\approx 4.2267 \end{aligned}}$$

Example 9-6d. Find the value of $\sqrt{\frac{25.8}{64.8}}$.

Solution: It was shown in Section 9-3 that $\log \sqrt{a} = \frac{1}{2} \log a$. Also, $\log \frac{x_1}{x_2} = \log x_1 - \log x_2$. Thus,

$$\log \sqrt{\frac{25.8}{64.8}} = \frac{1}{2} (\log 25.8 - \log 64.8)$$

$$\log 25.8 \approx 1.4116$$

$$\log 64.8 \approx 1.8116$$

$$\log \frac{25.8}{64.8} \approx -.4000$$

$$\log \sqrt{\frac{25.8}{64.8}} \approx \frac{1}{2}(-.4000) = -0.2000.$$

Again, we observe that the number -0.2000, being negative, cannot be regarded as the mantissa of a logarithm, because all mantissas are, by definition, non-negative numbers less than one. Moreover, we have no negative entries in our table of mantissas (9-5d). Therefore, we must write the number -0.2000 in standard form, where the decimal fraction part is positive, as we did in some of the exercises following Definition 9-5a.

We have $-0.2000 = -0.2000 + 1 - 1 = 0.8000 - 1$

$$\therefore \log \sqrt{\frac{25.8}{64.8}} \approx -1 + .8000$$

$$\text{and } \sqrt{\frac{25.8}{64.8}} \approx 0.6310$$

Example 9-6e. Find the value of $(0.08432)^5$.

Solution: It was shown in Section 8-3 that $\log (0.08432)^5 = 5 \log (0.08432)$. From the Table 8-5d and the rules for characteristics,

$$\log (0.08432) \approx -2 + .9259.$$

$$\log (0.08432)^5 \approx -10 + 4.6295$$

$$\approx -6 + 0.6295$$

Then $(0.08432)^5 \approx 0.000004261.$

Note that it is often advantageous to keep the decimal fraction part of the logarithm positive. For this reason we did not express log 0.08432 in the equivalent form -1.0741 although it would not have been wrong to do so. In fact, if we use this value, we have

$$\log (0.08432)^5 \approx -5.3705$$

which is correct, but, because the decimal fraction part is negative, is not useable with our table. If we add and subtract 6, we have

$$-5.3705 + 6 - 6 = 0.6295 - 6 \quad \text{as shown above.}$$

Example 9-6f. Find the value of $\sqrt[3]{(0.07846)^4}$

Solution: The calculation is carried out as follows:

$$\log \sqrt[3]{(0.07846)^4} = \frac{1}{3}\,[4 \log 0.07846]$$

$$\log 0.07846 \approx -2 + .8947 = -1.1053$$

$$4 \log 0.07846 \approx -4.4212$$

$$\frac{1}{3}\,[4 \log 0.07846] \approx -1.4737 = -1.4737 + 2 - 2$$

$$\approx .5263 - 2$$

$$\therefore \log \sqrt[3]{(0.07846)^4} \approx -2 + .5263$$

$$\sqrt[3]{(0.07846)^4} \approx 0.03360$$

Exercises 9-6.

Use Table 9-5d to compute the value of the unknown in each of the following expressions:

1. $x = 53.89 \times 0.7394$.
2. $y = (141.6)(0.299)$.
3. $s = \dfrac{98.43}{253.7}$.
4. $x = \dfrac{1111}{0.0007382}$.

5. $x = \frac{0.0693}{2.196}$.

6. $x = \frac{3.579 \times 10^{-4}}{9.753 \times 10^{4}}$.

7. $z = \frac{640 \times (0.849)}{31.4}$.

8. $y = (0.0315)^3$.

9. $x = (0.008976)^4$.

10. $t = (6.432)^3 \times (8.595)^4$.

11. $z = \frac{1}{(1.23)^6}$.

12. $x = \frac{(64.95)^3}{(8.954)^6}$.

13. $y = \sqrt[4]{0.03107}$.

14. $t = \sqrt[7]{(0.09562)^4}$.

15. If $d = \sqrt[3]{\frac{38H}{R}}$, find H when $d = 2.166$ and $R = 1200$.

16. If $t = \pi\sqrt{\frac{l}{g}}$, find t when $l = 95.8$, and $g = 980$. Use $\pi = 3.14$.

17. $x = \sqrt{\frac{0.07324 \times (232.8)^2}{(0.8954)^2 \times (735.7)^2}}$.

18. $x = \frac{\log 97}{\log 134.4}$.

19. $r = \sqrt{\frac{6321}{81.25\ \sqrt[3]{0.16}}}$.

20. $y = \frac{(6.385)^3 \times (8.438)^2}{\sqrt[3]{(0.6359)^5}}$.

21. $y = \frac{(6.385)^3 + (8.438)^2}{\sqrt[3]{(0.6359)^5}}$.

22. $x = \left[\dfrac{\sqrt{8453}\ \ (0.004954)}{695.8} \right]^{\frac{1}{2}}$.

23. $2 \log x + \log \left(\dfrac{2x}{5}\right) = 6$

24. If $A = (1 + r)^n$, find

(a) A when $n = 30$ and $r = 0.03$,

(b) r when $A = 3$ and $n = 40$,

(c) A when $r = -0.05$ and $n = -20$.

9-7 Logarithms with an Arbitrary Base.

In Section 9-1, there was defined for each $k > 0$ a logarithm function as the area associated with the hyperbola $y = \frac{k}{x}$. For particular values of k like $k = 1$ and $k = M = \frac{1}{\ln 10} = 0.43429\ldots$ (see 9-1a), we obtain the natural logarithm function and the common logarithm function respectively. In general, as stated in Equation 9-1, the logarithm function associated with a particular positive value of k has the property that

$$\log x = k \ln x, \quad x > 0.$$

If a is any positive number which is not equal to one, then the ratio

$$\frac{\log x}{\log a} = \frac{k \ln x}{k \ln a} = \frac{\ln x}{\ln a}$$

is independent of the particular k used to define the log function. In other words, the ratio of the values $\log x$ to $\log a$ depends only on the numbers x and a and not on the particular logarithm function used. Thus, the function f_a defined by

9-7a $$f_a(x) = \frac{\log x}{\log a}, \quad x > 0$$

is independent of k. For example

$$f_{10}(x) = \frac{\log_{10} x}{\log_{10} 10} = \log_{10} x,$$

since $\log_{10} 10 = 1$.

Definition 9-7a. For $a > 0$ and $a \neq 1$, the function f_a defined by (9-7a) is called the logarithm function with base a. We write $f_a(x)$ as $\log_a x$. Thus

9-7b $$\log_a x = \frac{\log x}{\log a}.$$

Hence, for each positive $a \neq 1$, we have associated a logarithm function with base a. In particular, the equation preceding Definition (9-7a) tells us that this new logarithm function with base 10 is our old friend the common logarithm function. If we denote that value x for which $\ln x = 1$ by the letter e, then, by Definition (9-7a), the logarithm function with base e is given by

$$\log_e x = \frac{\log x}{\log e}.$$

But since $\frac{\log x}{\log e} = \frac{\ln x}{\ln e}$ for any logarithm function,

9-7c $$\log_e x = \ln x.$$

That is, the natural logarithm function is precisely the logarithm function with base e. The number e therefore, takes on a special significance. It is an irrational number whose value, correct to 10 decimal places is given by

$$e \approx 2.7182818285.$$

Notice that logarithms with base 1 are excluded from Definition (9-7a) because $\log 1 = 0$.

The motivation for defining $\log_a x$ as that ratio $\frac{\log x}{\log a}$ is that this ratio depends only on x and a, and not on the particular positive number k used to define $\log x$. As a matter of fact, the ratio $\frac{\log_b x}{\log_b a}$ is independent of b. Note that

$$\log_b x = \frac{\log x}{\log b}, \text{ and } \log_b a = \frac{\log a}{\log b}$$

so that their ratio is precisely $\frac{\log x}{\log a}$ which, by definition is $\log_a x$.

This simple relation,

9-7d $$\log_a x = \frac{\log_b x}{\log_b a}$$

for positive a and b different from one, is called the change of base law for logarithms.

Two particular bases are interesting. Let $x = b$ in (9-7d); since $\log_b b = \frac{\log b}{\log b} = 1$, we have

9-7e $$\log_a b = \frac{1}{\log_b a}.$$

Again, if we let $b = \frac{1}{a}$ in (9-7d), then

(i) $$\log_a x = \frac{\log_{\frac{1}{a}} x}{\log_{\frac{1}{a}} a}$$

But $\log_{\frac{1}{a}} a = \frac{\log a}{\log \frac{1}{a}} = \frac{\log a}{-\log a} = -1$ and (i) becomes

9-7f $$\log_a x = -\log_{\frac{1}{a}} x.$$

We write down several simple properties of the logarithm functions with arbitrary bases. The proofs of these properties follow immediately from the fact that $\log_a x = \frac{\log x}{\log a}$. The proofs are left as exercises.

9-7g $$\log_a 1 = 0.$$

9-7h $$\log_a a = 1.$$

9-7i $$\log_a a^n = n, \text{ for any integer } n.$$

9-7j $$\log_a (x_1 \cdot x_2) = \log_a x_1 + \log_a x_2.$$

Two other properties which will play an important part in the next section are

9-7k $\log_a x_1 = \log_a x_2$ if and only if $x_1 = x_2$.

9-7ℓ For each real number s, the equation $\log_a x = s$ has a unique solution.

To prove (9-7k), we observe that

$$\log_a x_1 = \log_a x_2 \text{ or } \frac{\log x_1}{\log a} = \frac{\log x_2}{\log a} \text{ if and only}$$

if $\log x_1 = \log x_2$. Moreover, according to (9-3h) and (9-3h'), $\log x_1 = \log x_2$ if and only if $x_1 = x_2$ and our proof is complete.

To prove (9-7ℓ), we observe that $\log_a x = s$ is equivalent to the equation $\log x = s \log a$, which has a unique solution according to (9-4f) if we consider c to be the real number $s \log a$.

The following examples illustrate the applications of some of the relations developed in this section.

First, we compute some logarithms with various bases (Examples 9-7a to 9-7e).

Example 9-7a. Compute $\log_2 8$.

Solution: $\log_2 8 = \frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2} = \frac{3 \log 2}{\log 2} = 3$.

Example 9-7b. Compute $\log_4 32$.

Solution: $\log_4 32 = \frac{\log 32}{\log 4} = \frac{\log 2^5}{\log 2^2} = \frac{5 \log 2}{2 \log 2} = \frac{5}{2}$.

Example 9-7c. Compute $\log_5 10$.

Solution: This time we use (9-7d) instead of using (9-7b) as we did in the first two examples.

$$\log_5 10 = \frac{\log_{10}10}{\log_{10}5} = \frac{1}{\log_{10}5} \approx \frac{1}{.6990} \approx 1.4306.$$

Note that it is possible to avoid this long division by using logarithms. Let $t = \frac{1}{.6990}$. Then,

$$\begin{aligned} \log t &= -\log .6990 \approx -(-1 + .8445) \\ &\approx 1 - .8445 = .1555 \\ \therefore t &\approx 1.431. \end{aligned}$$

Example 9-7d. Compute $\log_{\frac{1}{5}} 10$.

Solution: $\log_{\frac{1}{5}} 10 = \frac{\log 10}{\log 5^{-1}} = \frac{\log 10}{-\log 5} = -\log_5 10 \approx -1.431.$

Of course this answer could have been obtained by applying (9-7f) to the results of Example 9-7c.

Example 9-7e. Find N if $\log_3 N = 4$.

Solution: $\log_3 N = \frac{\log N}{\log 3} = 4$

$$\begin{aligned} \log N &= 4 \log 3 = \log 3^4 = \log 81 \\ \therefore N &= 81. \end{aligned}$$

Example 9-7f. Show that $\log_a x^n = n \log_a x$ if n is an integer.

Solution: We know that $\log x^n = n \log x$ by (9-3e). By Definition 9-7a, $\log_a x^n = \frac{\log x^n}{\log a} = \frac{n \log x}{\log a}$

$$= n \frac{\log x}{\log a} = n \log_a x.$$

Example 9-7g. The logarithm function corresponding to $k = 2$ coincides with the logarithm function with what base?

Solution: We know that $\log_b x$ is that logarithm function whose value at b is 1. According to Equation 9-1,

$\log x = k \ln x$. Since $\log b = 1$, we have

$1 = 2 \ln b$ for $k = 2$.

$\therefore \ln b = \frac{1}{2}$. But, by (9-7c),

$$\ln b = \log_e b = \frac{\log b}{\log e}.$$

$$\therefore \frac{\log b}{\log e} = \frac{1}{2}$$

$\log b = \frac{1}{2} \log e = \log \sqrt{e}$ (9-3f, $p = 1$, $q = 2$)

$\therefore b = \sqrt{e}$, because $x_1 = x_2$ if $\log x_1 = \log x_2$.

Exercises 9-7a

1. Find the value of the following logarithms without the use of tables:

   (a) $\log_9 81$

   (b) $\log_{32} \frac{1}{4}$

   (c) $\log_{\frac{1}{4}} 32$

   (d) $\log_{27} \frac{1}{9}$

   (e) $\log_{\frac{2}{3}} \frac{9}{4}$

   (f) $\log_{1.5} \frac{8}{27}$

   (g) $\log_{\pi} 1$

   (h) $\log_{10} 0.01$

   (i) $\log_{\sqrt{2}} 8$

   (j) $\log_{49} (\sqrt{7})^5$

2. Find b, x, or N:

(a) $\log_b 5 = \frac{1}{2}$

(b) $\log_{27} 9 = x$

(c) $\log_9 N = \frac{1}{2}$

(d) $\log_{\sqrt{5}} N = -4$

(e) $\log_{\frac{1}{4}} 64 = x$

(f) $\log_b 9\sqrt{3} = 5$

(g) $\log_{\frac{1}{16}} N = -0.75$

(h) $\log_b \frac{27}{8} = 1.5$

3. Make use of Table 9-5d to compute the following logarithms correct to the nearest thousandth:

(a) $\log_3 17$

(b) $\log_7 200$

(c) $\log_{0.4} 10$

(d) $\log_{13} 5$

(e) $\log_2 10$

(f) $\log_5 0.086$

4. Show that:

(a) $\log_5 2 \times \log_2 5 = 1$

(b) $\log_5 2 + \log_{\frac{1}{5}} 2 = 0$

5. Solve for x:

(a) $\log_5 x = 1.17$

(b) $\log_{\frac{1}{5}} x = -0.301$

6. Prove the following statements:

(a) 9-7g $\log_a 1 = 0$.

(b) 9-7h $\log_a a = 1$.

(c) 9-7i $\log_a a^n = n$ for any integer n.

(d) 9-7j $\log_a x_1x_2 = \log_a x_1 + \log_a x_2$.

7. If $\log_x N = s$, and $\log_x b = t$, find $\log_b N$.

8. Complete the following table:

N	1	2	3	4	5	6	7	8	9	10
$\log_2$ N										

9. The logarithm function corresponding to $k = 5$ coincides with the logarithm function with what base?

10. Compare the results of the preceding exercise with Example 7 and find the base of the logarithm function coinciding with the logarithm function corresponding to any $k > 0$.

11. Show that the solution of the equation $\log_a x = s$ is the same as the solution of the equation $\log_b x = s \log_b a$ provided a and b are positive numbers not equal to one, and s is a real number.

Let us examine the graphs of several logarithm functions defined by $y = \log_a x$.

If $a = 10$, we have the familiar graph of the common logarithm function shown in Figure 9-11. If $a = 100$, we can sketch the graph of $\log_{100} x$ by comparing it with the graph of $\log_{10} x$. To do this, we let $a = 10$ and $b = 100$ in Equation 9-7d and write

$$\log_{100} x = \log_{100} 10 \times \log_{10} x.$$

Now, $\log_{100} 10 = \frac{1}{2}$, so we obtain $\log_{100} x = \frac{1}{2} \log_{10} x$. From this we see that every ordinate of the graph of $y = \log_{100} x$ is one-half the corresponding ordinate of the graph of $y = \log_{10} x$. Similarly, each ordinate of the graph of $y = \log_{\frac{1}{10}} x$ is the negative of the corresponding ordinate of the graph of $y = \log_{10} x$; and each ordinate of $y = \log_{\frac{1}{100}} x$ is the negative of the corresponding ordinate of the graph of $y = \log_{100} x$. All four of these graphs are sketched in Fig. 9-7a.

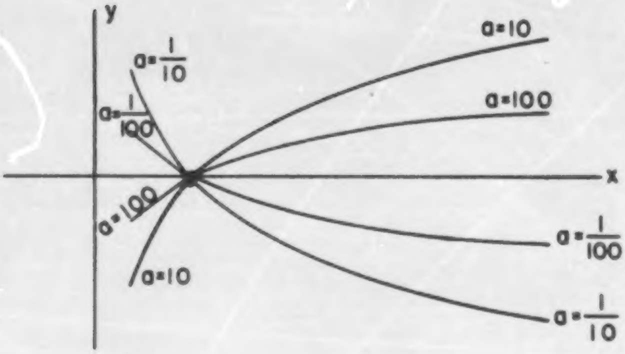

Fig. 9-7a

These graphs indicate that:

9-7m

(i) If $a > 1$, $\log_a x < 0$ if $x < 1$ and $\log_a x > 0$ if $x > 1$.

(ii) If $a < 1$, $\log_a x > 0$ if $x < 1$ and $\log_a x < 0$ if $x > 1$.

(iii) For $a > 1$, $\log_a x_1 < \log_a x_2$ if and only if $x_1 < x_2$.

(iv) For $a < 1$, $\log_a x_1 < \log_a x_2$ if and only if $x_1 > x_2$.

These statements are indeed true and they follow directly from (9-7b) and the corresponding properties of the log function given in Section 9-3.

Exercises 9-7b.

1. Sketch the graphs of $y = \log_{\sqrt{10}} x$ and $y = \log_{\frac{1}{\sqrt{10}}} x$ on the same set of axes.

2. If n is a natural number, show that each ordinate of the graph of $y = \log_{a^n} x$ is $\frac{1}{n}$ times the corresponding ordinate of the graph of $y = \log_a x$.

3. Prove:

   (a) If $1 < a$ and, $a < b$ then $\log_a b > 1$.

   (b) If $0 < a < 1$, and $a < b$ then $\log_a b < 1$.

4. Prove properties (i), (ii), (iii), (iv) for $\log_a x$ by making use of the corresponding properties of $\log x$ and without reliance on the graphs shown in Figure 9-7a.

5. Show that if $1 < a < b$ and $x > 1$ then $\log_a x > \log_b x$. If, however, $0 < x < 1$ then $\log_a x < \log_b x$.

The following is a summary of properties of logarithm functions with an arbitrary base:

Definition 9-7a. For $a > 0$ and $a \neq 1$, the logarithm function with base a is defined by the function

$$f_a(x) = \log_a x = \frac{\log x}{\log a}, \quad x > 0.$$

9-7c. $\log_e x = \ln x, \ x > 0.$

9-7d. $\log_a x = \dfrac{\log_b x}{\log_b a}, \ a \neq 1, \ b \neq 1, \ x > 0, \ a > 0, \ b >$

9-7e. $\log_a b = \dfrac{1}{\log_b a}, \ a \neq 1, \ b \neq 1, \ a > 0, \ b > 0.$

9-7f. $\log_a x = -\log_{\frac{1}{a}} x, \ a \neq 1, \ a > 0, \ x > 0.$

9-7g. $\log_a 1 = 0, \ a \neq 1, \ a > 0.$

9-7h. $\log_a a = 1$, $a \neq 1$, $a > 0$.

9-7i. $\log_a a^n = n$, for any integer n, $a \neq 1$, $a > 0$.

9-7j. $\log_a x_1 \cdot x_2 = \log_a x_1 + \log_a x_2$, $a \neq 1$, $a>0$, $x_1>0$, $x_2>0$.

9-7k. $\log_a x_1 = \log_a x_2$ if and only if $x_1 = x_2$, $a \neq 1$, $a>0$, $x_1>0$, $x_2>0$.

9-7ℓ. For each real number s, the equation $\log_a x = s$ has a unique solution, $a > 0$, $a \neq 1$.

9-7i' $\log_a x^n = n \log_a x$, for any integer n, $a \neq 1$, $a>0$, $x>0$.

9-7m $\log_a x_1 < \log_a x_2$ if and only if $x_1 < x_2$, $a \neq 1$, $a>0$, $x_1>0$.

9-8. Exponential Functions --Laws of Exponents

Let us look once again at the graph of the function defined by $y = \log_a x$, $(a > 0, a \neq 1, x > 0)$.

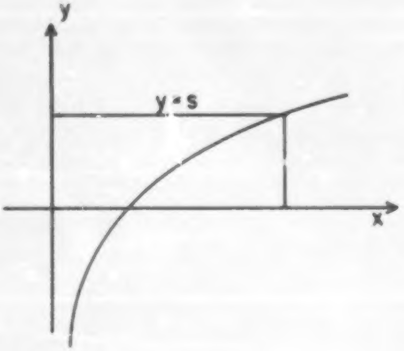

Fig. 9-8a

The domain of this function consists of all positive numbers and its range consists of all real numbers. We have seen that any horizontal line $y = s$ will intersect this graph in one and only one point (Figure 9-8a). In other words the equation $\log_a x = s$

has a unique solution. According to our discussion of inverse functions in Chapter 3, Section 8, the logarithm function has an inverse function which we will call, for the moment, E_a. This inverse function is then defined by the equation,

$$y = E_a(x).$$

We should note that $E_a(x)$ is not defined for $a = 0$ or 1 because $\log_a x$ is not defined for these values of a.

Again drawing on Chapter 3, we recall that inverse functions have the property that their graphs are symmetric in the line $y = x$. This fact enables us to sketch the graph of $y = E_a(x)$. This could be accomplished by drawing the graph of $y = \log_a x$ in ink and then folding the paper along the line $y = x$ so that an impression is made while the ink is still wet. The resulting graph of $y = E_a(x)$ is shown in Figure 9-8b.

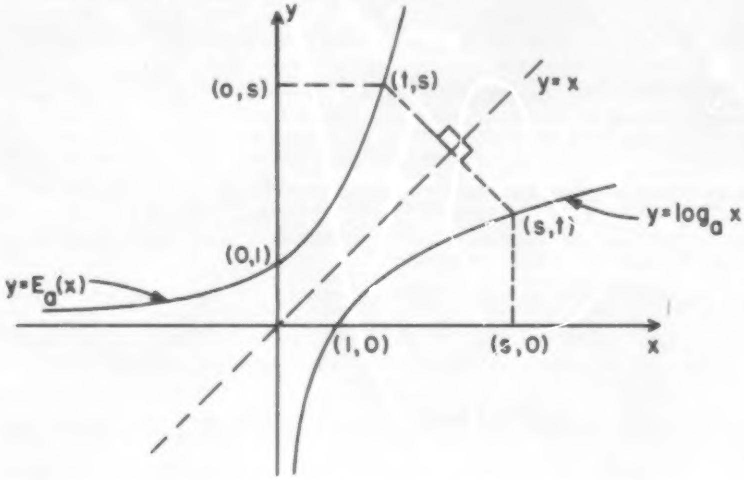

Figure 9-8b

From the graph it is clear that the domain of E_a is all real numbers and the range of E_a is all positive numbers.

Since E_a and $\log_a$ are inverse functions, we know from our discussion in Chapter 3, Section 8, that each of them "undoes" what the other one does. This means that

9-8a $E_a[\log_a s] = s$ or (i) $\log_a s$ is the unique solution of $E_a(x) = s$, and

9-8b $\log_a[E_a(u)] = u$ or (ii) $E_a(u)$ is the unique solution of $\log_a x = u$.

This latter fact, (ii), enables us to compute $E_a(n)$ when n is an integer. We ask for the solution of

$\log_a(x) = n$, where n is an integer.

Since E_a and $\log_a$ are inverse functions, we know that

$\log_a[E_a(n)] = n$. However, according to (9-7i),

$\log_a a^n = n$, where n is an integer. Therefore,

9-8c $E_a(n) = a^n$, because $x_1 = x_2$ if $\log_a x_1 = \log_a x_2$ according to (9-7k).

In particular, if $n = 0$, we have

9-8d. $E_a(0) = 1$, and if $n = 1$

9-8e. $E_a(1) = a$.

Equation 9-8c furnishes us with a compelling and permanent notation for the function E_a. The function is called the exponential function with base a, and $E_a(s)$ is written a^s, where a is called the base and s is called the exponent. The symbol a^s is read as "a to the s^{th} power", or simply "a to the s".

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Let us now review what we know about the function E_a:

(i) $E_a(s)$ is defined for all real numbers s.

(ii) $E_a(s)$ is the unique solution of $\log_a x = s$.

(iii) $E_a(s)$ has the same value as a^s when s is an integer n.

The first two statements follow directly from the fact that E_a was defined as the inverse of $\log_a$; the third statement is another way of saying Equation 9-8c.

These statements, (i), (ii), (iii) suggest that a^s might be defined in terms of $E_a(s)$, i.e. as a unique solution of the equation $\log_a x = s$. If this is done, we will have a serviceable definition for a^s when s is any real number, whereas until now a^s has been defined only for the case when s is an integer. Moreover, the new definition, while much broader, agrees with our previous interpretation of a^n.

Accordingly, we adopt the following definition:

Definition 9-8a. If $a > 0$, $a \neq 1$, and s is a real number, a^s is that real number x which is the unique positive solution of the equation $\log_a x = s$.

Since we now write a^s for $E_a(s)$, Equations 9-8a and 9-8b become respectively

9-8f $\quad a^{\log_a u} = u$ for all $u > 0$. ($\log_a u$ is the unique solution of $a^x = u$)

9-8g $\quad \log_a a^s = s$ for all real s. (a^s is the unique solution of $\log_a x = s$)

Equations (9-8f) and (9-8g) together are equivalent to this statement: If $a > 0$ and $a \neq 1$ then a^x and $\log_a x$ are inverse functions.

The meaning of our new definition (9-8a) and the equivalence of $E_a(s)$ and a^s are illustrated by the following examples and exercises:

Example 9-8a. Evaluate $3^{\frac{1}{3}}$.

Solution: According to our definition, $3^{\frac{1}{3}}$, or $E_3(\frac{1}{3})$, is the unique positive solution of the equation $\log_3 x = \frac{1}{3}$. But,

$$\log_3 x = \frac{\log_{10} x}{\log_{10} 3} . \qquad (9\text{-}7d)$$

$$\therefore \frac{\log_{10} x}{\log_{10} 3} = \frac{1}{3} \text{ or } \log_{10} x = \frac{1}{3}\log_{10} 3 \approx \frac{1}{3}(0.4771) \approx 0.1590,$$

and

$$x \approx 1.442 .$$

Example 9-8b. Use common logarithms to approximate the value of

$$3^{\sqrt{3}} .$$

Solution: $E_3(\sqrt{3})$ or $3^{\sqrt{3}}$ is defined as the positive solution of

$\log_3 x = \sqrt{3}$. Applying (9-7d) we have

$$\frac{\log_{10} x}{\log_{10} 3} = \sqrt{3} .$$

$$\therefore \quad \log_{10} x = \sqrt{3} \log_{10} 3 ,$$

$$\log_{10} x \approx 0.8267 ,$$

and

$$x \approx 6.710 .$$

Example 9-8c. Find the value of $2^{\frac{1}{2}}$ by sketching the graph of $E_2(x)$.

Solution: We seek $E_2(\frac{1}{2})$. We can obtain an approximate value from the graph of $E_2(x)$. We know that $E_2(x)$ is the inverse of the function $\log_2 x$. Therefore, we can obtain the graph of $E_2(x)$ by first graphing $y = \log_2 x$ and then reflecting this graph in the line $y = x$, as we did in Figure (9-8b). First, we make a table for $y = \log_2 x$.

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-4	-3	-2	-1	0	1	2	3

The corresponding table for $y = E_2(x)$ is obtained by interchanging x and y. Therefore, points $A(-4,\frac{1}{16})$, $B(-3,\frac{1}{8})$, $C(-2,\frac{1}{4})$, etc., lie on the graph of $y = E_2(x)$. Since, $\log_2 x$ has a continuous graph, its inverse function $E_2(x)$ also has a continuous graph. We obtain this graph by drawing a smooth curve through A, B, C, ... as shown in Figure (9-8c). The ordinate corresponding to $x = \frac{1}{2}$ is approximately 1.4.

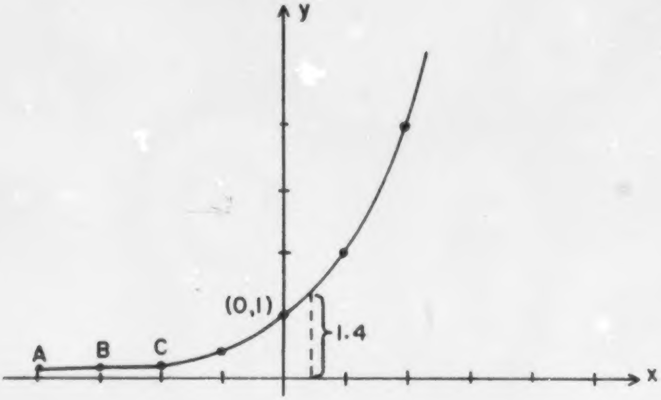


Figure 9-8c

Exercises 9-8a.

1. Evaluate the following by means of Definition 9-8a and Equations 9-8f and 9-8g:

(a) 5^3

(b) $4^{\frac{3}{2}}$

(c) $4^{-\frac{3}{2}}$

(d) $(1.5)^{-3}$

(e) $10^{2.4163}$

(f) $10^{0.2718 - 3}$

(g) $10^{-2.1871}$

(h) $10^{-1.4444}$

(i) $\log_{10} 10^{4.1623}$

(j) $\log_7 7^{2.43}$

(k) $7^{\log_7 0.0813}$

(ℓ) $5^{2 \log_5 3}$

2. Use common logarithms to find the approximate value of each of the expressions listed below. It will be necessary to make certain approximations and the answer you obtain will be only an approximate answer. This approximate answer should be as accurate as the use of a four place logarithm table will permit.

(a) $3^{1.72}$

(b) $4^{0.48}$

(c) $2^{\sqrt{2}}$

(d) $3^{-2.5}$

(e) $3^{\sqrt{2}}$

(f) $2^{\sqrt{3}}$

(g) $5^{\sqrt{3}}$

(h) $3^{-\sqrt{2}}$

(i) $(\sqrt{2})^{\sqrt{2}}$

(j) $(2.54)^{\sqrt{5}}$

(k) $10^{\sqrt{2}}$

(ℓ) $(1.25)^{-0.48}$

(m) 10^{π}

(n) $10^{1.54}$

(o) $10^{-0.42}$

(p) $(2.75)^{-3.2}$

3. Draw the graphs of the functions defined by

   (a) $y = 3^x$ (b) $y = \log_3 x$

   on the same set of axes. How are these graphs related? From the graphs read the approximate values of $3^{1.7}$ and $\log_3 1.7$.

4. Draw the graph of the function defined by $y = \log_{\sqrt{10}} x$. Write the equation which defines the inverse of this function. Draw the graph of the inverse function.

5. Follow the instructions given in Exercise 4 above for the functions defined by:

   (a) $y = \log_{\frac{1}{\sqrt{10}}} x$ (b) $y = (\frac{1}{2})^x$

6. Suppose C_1, C_2 and C_3 are the graphs of three functions, f_1, f_2, and f_3 and suppose further that (a) C_1 and C_2 are symmetric with respect to the y-axis, (b) C_2 and C_3 are symmetric with respect to $y = x$. If f_1 is defined by $y = (\frac{1}{a})^x$, $(a > 0, a \neq 1)$. Write the equation which defines f_3.

We are now in a position to prove a very important relation which is based on (9-8g) and two formulas from the preceding section, (9-7d) and (9-7e).

According to (9-7d)

$$\log_a x^t = \frac{\log_x x^t}{\log_x a} = \log_x x^t \cdot \frac{1}{\log_x a} .$$

But $\log_x x^t = t$ for t any real number by (9-8g),

and $\frac{1}{\log_x a} = \log_a x$ by (9-7e).

9-8h $\therefore \log_a x^t = t \log_a x.$

In dealing with positive integral exponents in our previous studies in algebra, the number a^s was interpreted to mean the product obtained when a is used as a factor s times. It was then a simple matter to verify that, if s and t are positive integers, and a and b are real numbers,

9-8i $$a^s \times a^t = a^{s+t}$$

9-8j $$(a^s)^t = a^{st}$$

9-8k $$(ab)^s = a^s b^s$$

Later we defined a^0 and a^{-s}, where s is a positive integer, so that (9-8i) remained valid, and we were led to conclude that

9-8ℓ $$a^0 = 1, \quad \text{and}$$

9-8m $$a^{-s} = \frac{1}{a^s}$$ are appropriate definitions provided $a \neq 0$. In fact, it is readily shown that (9-8i), (9-8j), and (9-8k) are valid when s and t are any integers if definitions (9-8ℓ) and (9-8m) are accepted.

Now we have assigned a meaning to a^s for any real exponent s, provided $a > 0$ and $a \neq 1$. Do the relations (9-8i) through (9-8m) remain valid when s and t are any real numbers and a and b any positive numbers not equal to one? The answer is yes, and we shall prove it directly. But let us first give a name to the relations (9-8i) through (9-8m) -- call them the laws of exponents. Moreover, to dispose of the case $a = 1$ which is not covered in our definition, let us agree that 1^s shall equal 1 for all real s. It is then easily seen that for $a = 1$ and $b > 0$ the laws of exponents are valid.

Theorem 9-8a. Let a and b be any positive numbers, then for all real numbers s and t the laws of exponents (9-8i) through (9-8m) are satisfied.

Proof of 9-8i: $a^s a^t = a^{s+t}$

(1) $\log_a a^s a^t = \log_a a^s + \log_a a^t$ (9-7j)

(2) $\log_a a^s = s$, $\log_a a^t = t$ (9-8f)

(3) $\log_a a^s a^t = s + t$ From (1) and (2)

$\log_a a^{s+t} = s + t$ (9-8f)

(4) $\log_a a^{s+t} = \log_a a^s a^t$ (3)

(5) $a^{s+t} = a^s a^t$ (9-7k)

Proof of 9-8j: $(a^s)^t = a^{st}$

(1) $\log_a (a^s)^t = t \log_a a^s$ (9-8h)

(2) $\log_a a^s = s$ (9-8f)

(3) $\therefore \log_a (a^s)^t = st$

(4) But, $\log_a a^{st} = st$ (9-8f)

(5) $\therefore \log_a (a^s)^t = \log_a a^{st}$ (3) and (4)

(6) $(a^s)^t = a^{st}$ (9-7k)

Proof of 9-8k: $(ab)^s = a^s b^s$

(1) $\log_{ab} (ab)^s = s$ (9-8f)

(2) $\log_{ab} a^s b^s = (\log_{ab} a^s) + (\log_{ab} b^s)$ (9-7j)

(3) $\log_{ab} a^s = s \log_{ab} a$ and

$\log_{ab} b^s = s \log_{ab} b$ (9-8h)

(4) $\therefore \log_{ab} a^s b^s = s(\log_{ab} a) + s(\log_{ab} b)$

$= s(\log_{ab} a + \log_{ab} b)$

(5) $\log_{ab} a + \log_{ab} b = \log_{ab} ab = 1$ (9-7k) and (9-7h)

(6) $\log_{ab} a^s b^s = s \times 1 = s$ (4) and (5)

(7) $\log_{ab}(ab)^s = \log_{ab} a^s b^s$ (1) and (6)

(8) $(ab)^s = a^s b^s$ (9-7k)

Proof of 9-8ℓ: $a^0 = 1$

(1) $\log_a 1 = 0$ (9-7g)

(2) $\log_a a^0 = 0$ (9-8f)(set $s = 0$)

(3) $a^0 = 1$ (9-7k)

Proof of 9-8m: $a^{-s} = \frac{1}{a^s}$

(1) $a^s a^{-s} = a^0 = 1$ (9-8i) and (9-8ℓ)

(2) But, $a^s \times \frac{1}{a^s} = 1$

(3) $\therefore\ a^{-s} = \frac{1}{a^s}$

In Section 7 we developed a "change of base" formula for the logarithm function. Equation (9-7d) can be written in the form

$$\log_b x = \log_a x \cdot \log_b a$$

which enables us to express the logarithm of x to the base b as a multiple of the logarithm of x to the base a. We now develop a similar change of base equation for the exponential function.

For example, we might ask: "What power of three is equal to the third power of nine?" To answer this we must solve the equation $3^x = 9^3$. In this case it is readily seen that the value of x is 6. Ordinarily the solution of the equation $a^x = b^s$ is more difficult.

We have learned that if two numbers are equal, then their logarithms to any base are equal (9-7k'). Therefore, $\log_a a^x = \log_a b^s$ and this equation is equivalent to $x = s \log_a b$ according to (9-7h) and (9-8h). Accordingly,

9-8n $$b^s = a^{s \log_a b}. \quad (a > 0,\ b > 0,\ s \text{ any real number}).$$

A special case of this formula which is frequently used in mathematics is obtained by letting $a = e$, the base of natural logarithms:

9-8o $$b^s = e^{s \log_e b} \quad \text{or} \quad b^s = e^{s \ln b}.$$

At this point it is appropriate for us to consider the relation between radical expressions such as

$$\sqrt{5},\quad \sqrt[3]{2},\quad \sqrt[q]{a}$$

and expressions involving positive rational exponents such as

$$5^{\frac{1}{2}},\quad 2^{\frac{1}{3}} \text{ and } a^{\frac{1}{q}}.$$

Consider first $\sqrt[q]{a}$ and $a^{\frac{1}{q}}$ where q is a natural number.

According to Definition 9-8a, $a^{\frac{1}{q}}$ is defined as the unique positive solution of $\log_a x = \frac{1}{q}$. That is,

(i) $$\log_a a^{\frac{1}{q}} = \frac{1}{q}.$$

In Section 3, $\sqrt[q]{a}$ is defined as the positive number whose q^{th} power is a. That is

(ii) $$(\sqrt[q]{a})^q = a\ .$$

If two positive numbers are equal then their logarithms to base a are equal (9-7k). Hence,

(iii) $\log_a (\sqrt[q]{a})^q = \log_a a$, or

(iv) $q \log_a \sqrt[q]{a} = 1$ (9-8h) and (9-7h)

(v) $\log_a \sqrt[q]{a} = \frac{1}{q}$

(vi) $\log_a \sqrt[q]{a} = \log_a a^{\frac{1}{q}}$ (i) and (v)

(vii) $\sqrt[q]{a} = a^{\frac{1}{q}}$. (9-7k)

We have thus established

9-8p $\sqrt[q]{a} = a^{\frac{1}{q}}$ where $a > 0$ and q is a natural number.

Now that we have established the equality of $\sqrt[q]{a}$ and $a^{\frac{1}{q}}$, it is readily seen that

9-8q $\sqrt[q]{x^p} = (\sqrt[q]{x})^p = x^{\frac{p}{q}}$ where p and q are positive integers.

The proof requires our new equality and the "power of a power" law, (9-8j). We have

(i) $\sqrt[q]{x^p} = (x^p)^{\frac{1}{q}} = x^{\frac{p}{q}}$ and

(ii) $(\sqrt[q]{x})^p = (x^{\frac{1}{q}})^p = x^{\frac{p}{q}}$.

Equations (i) and (ii) together are equivelent to (9-8q).

We close this section with the statement of a theorem which summarizes the relation between logarithms and exponents:

Theorem 9-8b. $a^s = N$ if and only if $s = \log_a N$, provided a is positive $\neq 1$ and s is real.

Proof:

(1)	$a^s = N$	Hypothesis
(2)	$\log_a a^s = \log_a N$	(9-7k)
(3)	But, $\log_a a^s = s$	(9-8g)
(4)	$\therefore s = \log_a N$	

The proof of the second part of this theorem, if $s = \log_a N$, then $a^s = N$, is left to the student.

The following is a summary of the properties of the exponential function:

Definition 9-8a. If $a > 0$, $a \neq 1$, and s is a real number, a^s is that real number x which is the unique positive solution of the equation $\log_a x = s$.

9-8f $a^{\log_a u} = u$ for all $u > 0$, $a > 0$, $a \neq 1$.

9-8g $\log_a a^s = s$ for all real s, $a > 0$, $a \neq 1$.

Equations (9-8f) and (9-8g) together are equivalent to this statement: If $a > 0$ and $a \neq 1$, then a^x and $\log_a x$ are inverse functions.

9-7h $\log_a x^t = t \log_a x$ for $a > 0$ and $a \neq 1$ and $x > 0$, t real.

9-8i $a^s \times a^t = a^{s+t}$, $a > 0$, s and t real.

9-8j $(a^s)^t = a^{st}$, $a > 0$, s and t real.

9-8k $(ab)^s = a^s b^s$, $a > 0$, $b > 0$, s real.

9-8ℓ $a^0 = 1$, $a > 0$.

9-8m. $a^{-s} = \frac{1}{a^s}$, $a > 0$, s real

9-8n $b^s = a^{s \log_a b}$, $a > 0$, $b > 0$, $a \neq 1$, s real

9-8o $b^s = e^{s \log_e b} = e^{s \ln b}$, $b > 0$, s real.

9-8q $\sqrt[q]{x^p} = x^{\frac{p}{q}}$, $x > 0$, p and q are positive integers.

Theorem 9-8b $a^s = N$ if and only if $s = \log_a N$.

The following examples show some applications of the laws of exponents:

Example 9-8d. Show that $(\frac{a}{b})^s = \frac{a^s}{b^s}$ where a and b are positive real numbers, and s is real.

First Solution:

$$\log_c (\tfrac{a}{b})^s = s \log_c (\tfrac{a}{b}) = s(\log_c a - \log_c b)$$

Also,
$$\log_c \frac{a^s}{b^s} = \log_c a^s - \log_c b^s = s \log_c a - s \log_c b = s(\log_c a - \log_c b)$$

$\therefore (\frac{a}{b})^s = \frac{a^s}{b^s}$ because $x_1 = x_2$ if $\log_c x_1 = \log_c x_2$.

Second Solution: We learned in Chapter 1, Section 6, that $\frac{a}{b} = a \times \frac{1}{b}$. Moreover, $\frac{1}{b}$ is written as b^{-1}.

$$\therefore (\tfrac{a}{b})^s = (a \times \tfrac{1}{b})^s = (a \times b^{-1})^s = a^s \times b^{-s} = \frac{a^s}{b^s} .$$

Example 9-8e. Expressions involving radicals may be expressed in equivalent forms involving positive rational exponents. Verify the following:

(a) $\sqrt[3]{\sqrt[5]{a}} = (a^{\frac{1}{5}})^{\frac{1}{3}} = a^{\frac{1}{15}}$. (9-8q and 9-8j)

(b) $\left(\sqrt[7]{a} \cdot \frac{1}{\sqrt[3]{b^4}}\right)^{\frac{1}{5}} = \left(a^{\frac{1}{7}} \cdot \frac{1}{b^{\frac{4}{3}}}\right)^{\frac{1}{5}} = \frac{a^{\frac{1}{35}}}{b^{\frac{4}{15}}}$.

Example 9-8f. An expression involving rational exponents may be converted into an equivalent radical form. Verify the following:

(a) $3^{\frac{1}{4}} \times 3^{\frac{1}{5}} = 3^{\frac{5}{20}} \times 3^{\frac{4}{20}} = 3^{\frac{5}{20} + \frac{4}{20}} = 3^{\frac{9}{20}} = \sqrt[20]{3^9}$.

(b) $(7^{-3})^{-\frac{3}{4}} = 7^{\frac{9}{4}} = 7^{2 + \frac{1}{4}} = 49\sqrt[4]{7}$.

(c) $a^{\frac{1}{4}} \div a^{\frac{2}{3}} = a^{\frac{1}{4}} \times \frac{1}{a^{\frac{2}{3}}} = a^{\frac{1}{4}} \times a^{-\frac{2}{3}} = a^{\frac{1}{4} + (-\frac{2}{3})}$

$$= a^{\frac{3}{12} - \frac{8}{12}} = a^{-\frac{5}{12}} = \frac{1}{a^{\frac{5}{12}}} = \frac{1}{\sqrt[12]{a^5}} .$$

Example 9-8g. Expressions involving negative integral exponents may be changed to equivalent expressions in which all exponents are positive. Verify the following:

(a) $(\frac{t}{a})^{-3} = \frac{1}{(\frac{t}{a})^3} = \frac{1}{\frac{t^3}{a^3}} = \frac{a^3}{t^3}$.

(b) $\frac{a^{-1}\ b^{-1}}{a^{-1} + b^{-1}} = \frac{(a^{-1}b^{-1})\ ab}{(a^{-1} + b^{-1})\ ab} = \frac{1}{b + a}$.

Example 9-8h. Some expressions involving radicals and rational exponents are easily evaluated. Verify the following:

(a) $49^{\frac{1}{2}} = \sqrt{49} = 7$.

(b) $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$.

(c) $(\sqrt[3]{7})^6 = (7^{\frac{1}{3}})^6 = 7^2 = 49$.

(d) $(2^{-4})^{-3} = 2^{12} = 4096$.

Example 9-8i. Express $7^{4.13}$ as a power of 13.

Solution: We apply (9-8n), $b^s = a^{s \log_a b}$. Let $b = 7$, $s = 4.13$ and $a = 13$. We have $7^{4.13} = 13^{4.13 \log_{13} 7} = 13^x$ where $x = 4.13 \log_{13} 7$. $\log_{13} 7 = \frac{\log_{10} 7}{\log_{10} 13}$ (9-7d)

and
$$\log_{13} 7 \approx \frac{0.8451}{1.1139}$$

$$\therefore\ 4.13 \log_{13} 7 \approx \frac{4.13 \times 0.8451}{1.114} = x$$

and
$$\log x = \log 4.13 + \log 0.8451 - \log 1.114\ .$$

$$\therefore\quad x = 3.134$$

and
$$7^{4.13} \approx 13^{3.134}$$

Example 9-8j. Equations in which at least one exponent is a function of the unknown are called exponential equations.

(a) $3^{x^2 - 6} = 9^{\frac{1}{2}x}$

(b) $3^{x + \frac{1}{2}} = 5^x$

Solution:

(a) $9^{\frac{1}{2}x} = (3^2)^{\frac{1}{2}x} = 3^x$ $\therefore 3^{x^2 - 6} = 3^x$. If we find the logarithm of each member, we obtain

$x^2 - 6 = x$, or

$x^2 - x - 6 = 0$.

$\therefore x = 3$ or $x = -2$. Each result checks.

(b) Find the common logarithm of each member:

$$\log_{10} 3^{x + \frac{1}{2}} = \log_{10} 5^x$$

$$(x + \tfrac{1}{2}) \log_{10} 3 = x \log_{10} 5$$

$$(x + \tfrac{1}{2})(0.4771) \approx x(0.6990)$$

$$x \approx 1.075 \ .$$

Exercises 9-8b.

1. Evaluate the following.

(a) $27^{\frac{1}{3}}$

(b) 11^0

(c) 5^{-1}

(d) $81^{\frac{3}{4}}$

(e) $81^{-\frac{3}{4}}$

(f) $(\frac{5}{6})^{-2}$

(g) $(0.027)^{\frac{1}{3}}$

(h) $(0.0001)^{\frac{1}{4}}$

(i) $(\frac{1}{4})^{-\frac{5}{2}}$

(j) $(\frac{9}{4})^{-\frac{3}{2}}$

(k) $(3^{-2})^{-3}$

(ℓ) $\sqrt[6]{27}$

(m) $(\sqrt[3]{7})^0$

(n) $23^{\frac{1}{3}} \cdot 23^{\log_8 4}$

(o) $[(0.2)^{\frac{1}{2}}]^{-4}$

(p) $(5^{\sqrt{3}})^{\sqrt{3}}$

(q) $7^{1.26} \times 7^{1.74}$

(r) $(5^{\sqrt{2}} \cdot 7^{\sqrt{2}})^{\sqrt{2}}$

(s) $\left(\pi^{\sqrt[3]{7}}\right)^0$

2. Write each of the following as an equivalent expression in which all exponents are positive.

(a) $(ab)^{-1}$

(b) $(a^{-2}b)^{-3}$

(c) $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$

(d) a^2b^{-1}

(e) $(x^{-1} + y^{-1})^2$

(f) $\dfrac{3xy}{x^{-1} + y^{-1}}$

(g) $\dfrac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}}$

(h) $\dfrac{x^{-3} + y^{-3}}{x^{-1} + y^{-1}}$

(i) $(x^{-2} + y)^{-2}$

3. Write each of the following as an equivalent expression for (a) - (i) in radical form.

(a) $z^{\frac{1}{6}}$

(b) $ax^{\frac{1}{3}}$

(c) $(a^{\frac{2}{3}})^{\frac{1}{7}}$

(d) $(11^{-2})^{-\frac{2}{3}}$

(e) $a^{\frac{1}{3}} + a^{\frac{1}{5}}$

(f) $5^{\frac{1}{3}} \cdot 5^{\frac{1}{7}}$

(g) $(\frac{a}{b})^{\frac{2}{3}}$

(h) $\left(\dfrac{c^{\frac{1}{2}}}{d^{-\frac{2}{3}}}\right)^{-6}$

(i) $\left(\dfrac{a^{\frac{1}{2}}\,x^{\frac{1}{3}}}{a^{-1} \cdot x^{\frac{1}{2}}}\right)^2$

*(j) Express with a single radical sign.

$$\left[(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})\right]^{\frac{1}{2}}$$

4. Use common logarithms to compute the value of each of the following expressions.

(a) $(24.80)^{-3}$

(b) $(276.3)^{-\frac{1}{5}}$

(c) $(0.8412)^{-5}$

(d) $(356.8)^{-1.1}$

(e) $\dfrac{(0.06381)^{-\frac{1}{4}}}{(0.9816)^{-1.1}}$

(f) $16\,(0.0071234)^{\frac{1}{3}}\,(82671)^{-\frac{1}{3}}$

5. Show that $x^s = y^s$ if and only if $x = y$ provided x and y are positive $\neq 1$ and s is real.

6. Solve the following equations for x.

(a) $2^{x + 6} = 32$

(b) $9^{2x} = 27^{3x - 4}$

(c) $2^{x^2 + 4x} = \frac{1}{8}$

(d) $25^{2x} = 5^{x^2-12}$

(e) $8 = 4^{x^2} \cdot 2^{5x}$

(f) $2^x \cdot 5 = 10^x$

7. Solve the following equations for x.

(a) $10^{2x - 3} = 43$

(b) $e^{3x} = 16$

(c) $2^{3x} = 3^{2x + 1}$

(d) $5^{x + 2} = 7^{x - 2}$

(e) $(1.03)^x = 2.500$

(f) $e^{2x} - 2e^x + 1 = 0$

(g) $\log_3 (x + 1) + \log_3 (x+3) = 1$

(h) $\dfrac{\log_{10} (7x - 12)}{\log_{10} x} = 2$

(i) $\log_5 \sqrt{\dfrac{3x + 4}{x}} = 0$

8. Prove that x^s and $x^{\frac{1}{s}}$ are inverse functions. $(x > 0, s \neq 0,$ s real). Graph $y = x^3$ and $y = x^{\frac{1}{3}}$ on the same set of axes.

9. Prove $a^s = N$ if $s = \log_a N$. $(a > 0,\ a \neq 1,\ N$ real, $N > 0.)$

10. Draw the graphs of (1) $y = 2^x$ and (2) $y = 4^x$ on the same set of axes. It will be found that each abscissa on the graph of (1) is twice the corresponding abscissa on the graph of (2). Can you generalize this statement for the graphs of (1) $y = a^x$ and (2) $y = b^x$ $(a>0,\ b>0,\ x$ real).

11. If $0 < a < b$ compare a^x and b^x when

    (a) $x > 0$

    (b) $x = 0$

    (c) $x < 0$.

12. Solve each of the following equations as indicated.

    (a) Solve $y = c\,x^n$ for n.

    (b) Solve $u = a\,e^{-bv}$ for v. (Use natural logarithms.)

    (c) Solve $s = a\left(\frac{1 - r^n}{1 - r}\right)$ for n.

    (d) Solve $x = \log_5 y$ for y.

    (e) Solve $\ell = ar^{n-1}$ for n.

    (f) Solve for x: $\log_{10}(x - 4) + \log_{10}(x + 3) = \log_{10}30$.

    (g) Solve the following equations for x and y.

$$\begin{cases} \log_{10}x + \log_{10}y = 2 \\ x + y = 25. \end{cases}$$

13. Solve for x.

    (a) $e^x + e^{-x} = 2$

    (b) $e^x - e^{-x} = 2$

14. According to the law of radioactive decay, the mass m remaining t years from now is given by the formula:

    $m = m_o e^{-ct}$, where e is the base of the natural logarithm, m_o is the present mass and c is a constant depending on the particular radioactive substance involved.

    The half-life of a radioactive substance is the time elapsed when $m = \frac{1}{2}m_o$. Find the half-life of a radioactive substance for which $c = 2$.

15. If an amount of money P is invested at an interest rate of r (expressed as a decimal) per year, the amount A accumulated at the end of n years, when interest is compounded annually, is given by the formula: $A = P(1 + r)^n$. This statement is known as the compound interest law.

    (a) Find the amount A to which an investment of \$1,000 will accumulate in 20 years if interst is compounded annually at 6%.

    (b) How long will it take for an investment to double itself if interest is compounded annually at $4\frac{1}{2}$%?

    (c) If one dollar grows to 3 dollars in 30 years when interest is compounded annually, find the approximate rate of interest.

9-9 Miscellaneous Exercises

1. Find the value of:

(a) $$\frac{\log_{25}5 - \log_{\frac{1}{16}} 8}{\log_9 \frac{1}{27} + \log_4 1}$$

(b) $\log_5 1 + \log_8 4 \cdot \sqrt[5]{16}$

(c) $$\frac{\log_{49}7 + \log_{27}9}{\log_{\frac{1}{32}} 64 - \log_{\frac{3}{2}} \frac{4}{9}}$$

(d) $$\frac{\log_3 81 - \log_\pi 1}{\log_{2\sqrt{2}} 8 - \log_{10} 0.001}$$

(e) $$\frac{\log_{216}6^4 - \log_{27}9^{\frac{3}{4}}}{\log_{\sqrt{3}} 1 + \log_4 8^{-\frac{3}{2}}}$$

(f) $$\frac{\log_9 81^{-\frac{3}{8}}}{\log_{49}7^{\frac{2}{3}} - \log_{64} \sqrt[5]{\frac{1}{16}}}$$

2. Find the value of x:

(a) $x = \log_3 81$

(b) $\log_2 x = -5$

(c) $\log_x 8 = \frac{3}{2}$

(d) $\log_5 0.2 = x$

(e) $\log_x 49 = 4$

(f) $\log_{\frac{1}{2}} 8 = x$

(g) $\log_9 x = -\frac{5}{2}$

(h) $\log_x 0.04 = -2$

(i) $\log_x \sqrt{6} = \frac{1}{4}$

(j) $\log_{0.1} 10 = x$

3. Complete the following statements:

(a) $\log_a a^f =$

(b) $a^{\log_a x} =$

(c) $\log_b x \cdot \log_a b =$

(d) $\log_b a \cdot \log_a b =$

(e) The inverse of the function defined by the equation $y = a^x$ is defined by the equation $y = ?$

4. Solve for x:

(a) $\log_5 x = 3$

(b) $\log_7 \sqrt{\frac{2x - 3}{x}} = 0$

(c) $\log_x 27 = \frac{3}{2}$

(d) $\log_{10} x^2 - \log_{10} x = 2$

(e) $e^{x \log_e b} \cdot e^{x \log_e c} = (bc)^2$

5. Write with positive exponents and simplify:

(a) $\frac{a^{-1}}{b^{-1}}$

(b) $(\frac{a}{b})^{-k}$

(c) $\frac{r^2}{s^{-1}}$

(d) $\frac{cd^{-3}}{4a^{-2}}$

(e) $\frac{x^3 y^3}{x^{-2} y^{-2}}$

(f) $\left(d^7 e^3\right)^{-3}$

(g) $\frac{2^{-2} + 3^{-2}}{2^{-2} \cdot 3^{-2}}$

(h) $x^{-1} + y^{-1}$

(i) $\frac{a^{-1} + b^{-1}}{(cd)^{-1}}$

(j) $\frac{a^{-2} + b^{-2}}{a^{-1} + b^{-1}}$

6. Solve for x. (N and a are positive real numbers).

(a) $a^x = N$,

(b) $x^a = N$, $x > 0$,

(c) $N = \log_a x$. $a \neq 1$, $x > 0$

7. Find the value of x by means of the laws of exponents:

(a) $2^{\frac{1}{2}} \cdot 2^{\frac{3}{4}} = 2^x$

(b) $3^x = 3^{-0.3} \cdot \sqrt[4]{3}$

(c) $64^{-\frac{2}{3}} = 2^x$

(d) $\dfrac{4^{\frac{1}{2}}}{\sqrt[3]{8^2}} = 2^x$

(e) $x = 16^{\frac{3}{4}} - 4^0 + \left(\frac{9}{4}\right)^{-\frac{3}{2}}$

(f) $\sqrt[3]{\dfrac{t}{\sqrt{t}}} = t^x$

8. Find the numerical value of x for each of the following. Base necessary computations on Table 9-5d.

(a) $10^x = 41.63$

(b) $3^x = 733$

(c) $x^5 = 972$

(d) $x^{\frac{3}{2}} = 400$

(e) $e^x = 35$

(f) $x = (4.17)^{0.52}$

(g) $\log_7 x = 2.4$

(h) $\log_7 700 = x$

(i) $x = 3^{-3.7}$

(j) $5^x = 0.083$

(k) $\log_{0.5} 0.03 = x$

(ℓ) $2.13^x = 0.0417$

(m) $(0.5)^x = 70$

9. Solve for x. Assume that all other letters represent positive real numbers. Express all logarithms to base 10.

(a) $a^x = b$

(b) $x^a = b$

(c) $b = \log_a x$

(d) $m = ax^n$

(e) $m = ar^x$

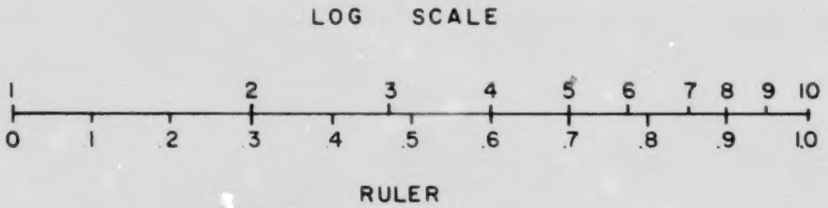
(f) $A = P(1 + x)^s$

(g) $A = P(1 + r)^x$

(h) $e^{x \ln b} = c$

*10. Compare the graphs of $y = \log_{10} 2x$ and $y = 10^{2x}$. Are these inverse functions?

*11. How to construct a logarithmic scale: On a sheet of paper draw a line 10 inches long. Mark off in tenths and hundredths. Fold the sheet of paper on the line so you can use it as a ruler. On another sheet of paper draw a line equal to 10 inches. Align the ruler with this line. Because the log of 1 is 0, place 1 on your new scale opposite 0 on you ruler. Because 0.3010 is the log of 2, place 2 opposite 0.30 on your ruler. Proceed similarly until you have placed 10 opposite 1 on your ruler.



Questions:

(a) Using two of these log scales, can you make a slide rule? Can you explain how a slide rule multiplies and divides?

(b) Construct a coordinate system using logarithmic scales on the coordinate axes instead of the normal linear scales. On this coordinate system plot $y = x^2$. Can you explain the result?

(c) Construct a coordinate system using a logarithmic scale on the axes of ordinates and the normal linear scale on the axis of abscissas. On this system plot $y = 2^x$. Can you explain the result?

Chapter 10

INTRODUCTION TO TRIGONOMETRY

10-1. Arcs and Paths.

Let P and Q be any two distinct points on a circle with radius r. These two points separate the circumference of a circle into two arcs, the sum of whose lengths is $2\pi r$. The length of any arc of a circle is equal to or less than $2\pi r$.

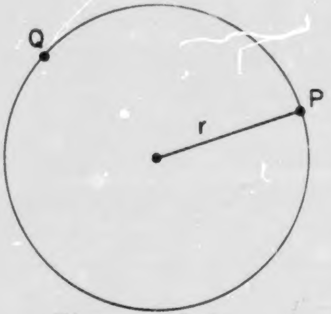


Figure 10-1a.
Arcs on a circle.

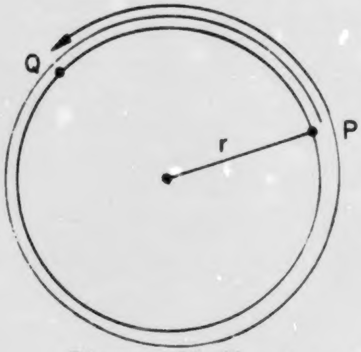


Figure 10-1b.
A path on a circle.

Let P be a point on a given circle as in Figure 10-1b. Let a point R start at P and move, without reversing its direction, a distance d along the circle to a final position Q (observe that d may be greater than the circumference of the circle). This motion will be called a path, and it will be denoted by the symbol (P,+d) if the motion is in the counter-clockwise direction, and by the symbol (P,-d) if the motion is in the clockwise direction around the circle. The symbol (P,0) corresponds to the path in which Q does not move from P. The point P is the initial point of of the path, and the final position of Q is the terminal point of the path.

Observe that every path is described by a symbol (P,c), where c is some positive or negative real number. Conversely, if c is any real number, there is a unique path on the given circle corresponding to the symbol (P,c).

Two paths (P_1,c_1) and (P_2,c_2) are equal if and only if $P_1 = P_2$ and $c_1 = c_2$. Two paths are equivalent if and only if $c_1 = c_2$. Observe that two paths are equivalent if they are equal, but that two equivalent paths need not be equal.

If (P,c) is any path on a given circle, there is a unique path (P_0,c) on this circle which has its initial point at a given point P_0 and which is equivalent to (P,c).

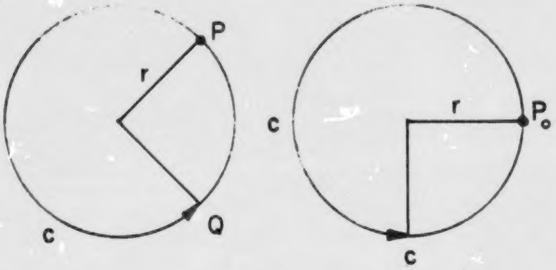


Figure 10-1c.
The unique equivalent path with initial point at P_0.

We shall now define the addition of paths. If (P_1,c_1) and (P_2,c_2) are any two paths on the same circle, then

$$(P_1,c_1) + (P_2,c_2) = (P_1,c_1 + c_2).$$

Since $(P_2,c_2) + (P_1,c_1) = (P_2,c_2 + c_1)$, we see that $(P_1,c_1) + (P_2,c_2)$ and $(P_2,c_2) + (P_1,c_1)$ are not equal unless $P_1 = P_2$. Nevertheless, $(P_2,c_2) + (P_1,c_1)$ is equivalent to $(P_1,c_1) + (P_2,c_2)$, since $c_2 + c_1 = c_1 + c_2$ by the commutative property of the addition of real numbers.

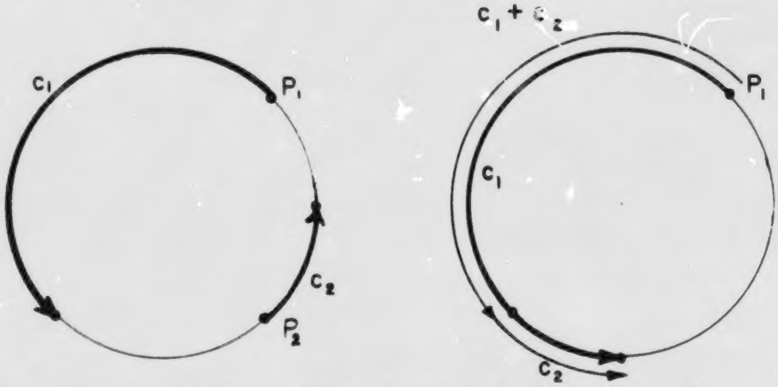


Figure 10-1d. Graph of $(P_1,c_1) + (P_2,c_2)$.

Exercises 10-1

1. Let P_1 and P_2 be points on a circle of radius 5 as shown in Figure 10-1e. Draw diagrams to show the following paths:

   (a) (P_1, π)

   (b) $(P_2, 10\pi)$

   (c) $(P_1, -\pi)$

   (d) $(P_2, -10\pi)$

   (e) $(P_1, 25\pi)$

   (f) $(P_1, -25\pi)$

   (g) $(P_2, 30\pi)$

   (h) $(P_2, -30\pi)$

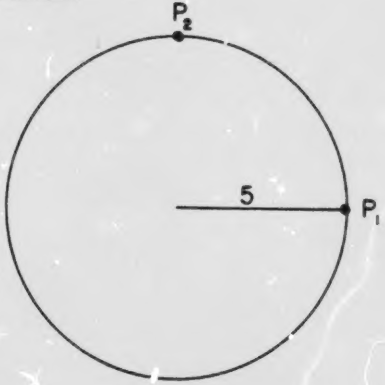


Figure 10-1e.
Figure for Exercises 10-1.

2. Draw diagrams to illustrate the following additions of paths:

   (a) $(P_1, \frac{5\pi}{2}) + (P_2, \frac{5\pi}{2})$

   (b) $(P_2, \frac{5\pi}{2}) + (P_1, \frac{5\pi}{2})$

   (c) $(P_1, 10\pi) + (P_2, -5\pi)$

   (d) $(P_2, -5\pi) + (P_1, 10\pi)$

   (e) $(P_1, \frac{15\pi}{2}) + (P_2, -\frac{5\pi}{2})$

   (f) $(P_2, -\frac{5\pi}{2}) + (P_1, \frac{15\pi}{2})$

3. Which ones of the sums in Exercise 2 above are equivalent?

10-2. Signed Angles.

The rays $\overrightarrow{AP}$ and $\overrightarrow{AQ}$ form an angle in the elementary sense (see Figure 10-2a). It will now be shown that paths on a circle can be used to extend the elementary notion of angle.

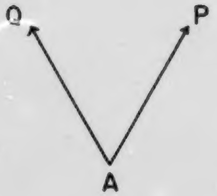


Figure 10-2a.
An angle in the elementary sense.

Consider a circle with radius 1 whose center is the vertex of the angle formed by rays $\overrightarrow{AP}$ and $\overrightarrow{AQ}$. Figure 10-2b shows a path (P,θ) on this circle that can be associated with this angle. It is immediately clear, however, that other paths could be associated with the elementary angle. To overcome this difficulty we introduced the notion of signed angle. There is a one-to-one correspondence between the set of signed angles and the set of paths on the unit circle.

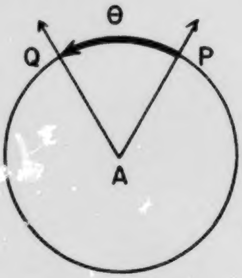


Figure 10-2b.
Angles and paths.

Let a path (P,θ) on the unit circle be given (see Figure 10-2c). The ray AP is the initial side of the corresponding signed angle. If Q is the terminal point of the path (P,θ), the ray $\overrightarrow{AQ}$ is the terminal side of the signed angle. The path (P,θ) specifies how the signed angle is generated, in the following sense. The ray $\overrightarrow{AR}$ is placed in the initial position $\overrightarrow{AP}$ and then rotated about A so that R traces the path (P,θ). The terminal position of AR is then $\overrightarrow{AQ}$. The signed angle is the triple $\{\overrightarrow{AP}, \overrightarrow{AQ}, (P,\theta)\}$; it is completely determined by the

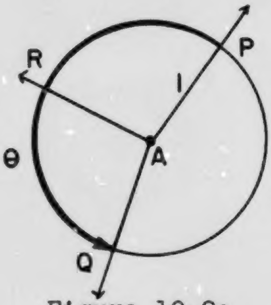


Figure 10-2c.
Generation of angles.

path (P,θ) and the vertex A in the sense that the signed angle can be constructed when A and (P,θ) are given. It is thus appropriate to denote the signed angle $\{AP, AQ, (P,\theta)\}$ by (A,P,θ).

A signed angle has a direction associated with it. If $\theta > 0$, the angle (A,P,θ) is generated by rotating AR in the counter-clockwise direction, and we say that the angle is positive; if $\theta < 0$, the angle (A,P,θ) is generated by rotating AR in the clockwise direction, and we say that the angle is negative.

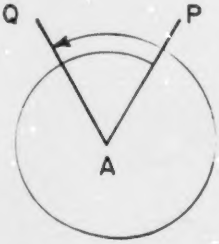
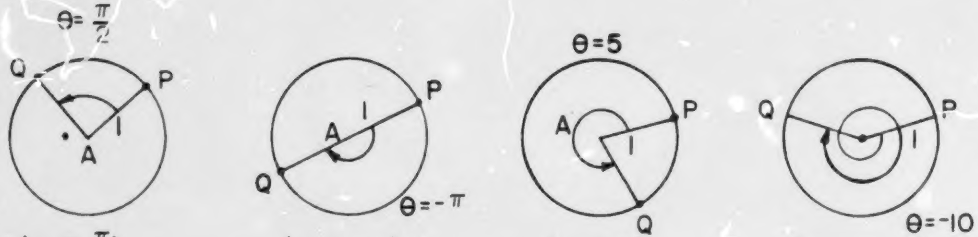


Figure 10-2d.
Signed angle.

Example 10-2a. Construct the angles $(A,P,\frac{\pi}{2})$, $(A,P,-\pi)$, $(A,P,5)$, and $(A,P,-10)$, where P is a given fixed point on a unit circle with center A.

Solution: The angles are shown in Figure 10-2e.



$(A,P,\frac{\pi}{2})$ $(A,P,-\pi)$ $(A,P,5)$ $(A,P,-10)$

Figure 10-2e. Construction of the four angles indicated.

We say that two angles (A_1,P_1,θ_1) and (A_2,P_2,θ_2) are equal if and only if $A_1 = A_2$, $P_1 = P_2$, and $\theta_1 = \theta_2$. If two angles are equal, they clearly have the same vertices, the same initial sides, and the same terminal sides. It is not true, however, that two angles with the same vertices and initial and terminal sides are equal. If (A_1,P_1,θ_1) and (A_2,P_2,θ_2) have the same initial and terminal sides, then $A_1 = A_2$ and

10-2a. $$\theta_1 = \theta_2 + 2n\pi,$$

where n is 0 or a positive or negative integer. Furthermore, angles with the same initial and terminal sides are called co-terminal angles. Two co-terminal angles are shown in Figure 10-2f. Two angles (A_1,P_1,θ_1) and (A_2,P_2,θ_2) are equivalent if and only if $\theta_1 = \theta_2$. If the signed angles (A_1,P_1,θ_1) and (A_2,P_2,θ_2) are equivalent, then the geometric angles $P_1A_1Q_1$ and $P_2A_2Q_2$ are congruent in the sense of geometry.

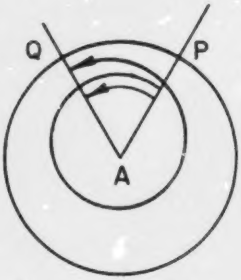


Figure 10-2f.
Two co-terminal angles.

Example 10-2b. The two angles $(A_1,P_1,\frac{\pi}{2})$ and $(A_2,P_2,\frac{\pi}{2})$ shown in Figure 10-2g are equivalent, but the two angles $(A_1,P_1,\frac{\pi}{2})$ and $(A_2,P_2,-\frac{\pi}{2})$ shown in Figure 10-2h are not equivalent.

An angle is said to be in standard position in a coordinate system if and only if its vertex is at the origin and its initial side extends along the positive x-axis. Every angle is equivalent to one and only one angle in standard position. It will be convenient to denote an angle in standard position by $(0,X,\theta)$.

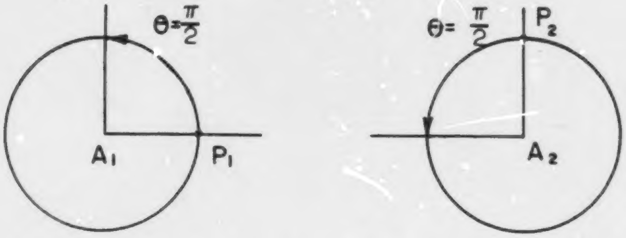

$\theta=\frac{\pi}{2}$
A_1
P_1
$\theta=\frac{\pi}{2}$
P_2
A_2

Figure 10-2g. Equivalent angles.

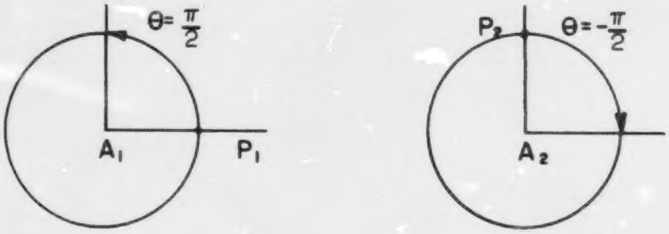
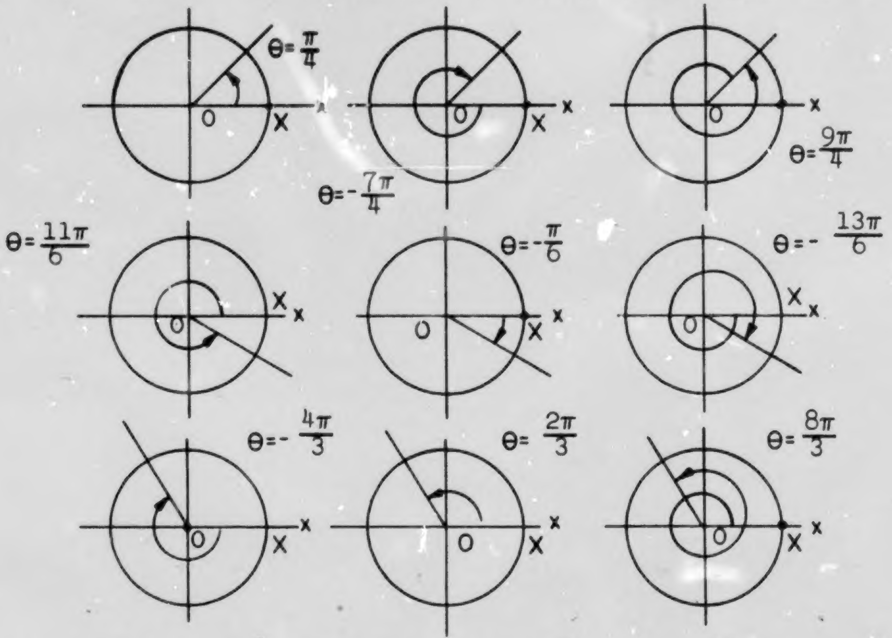

$\theta=\frac{\pi}{2}$
A_1
P_1
P_2
$\theta=-\frac{\pi}{2}$
A_2

Figure 10-2h. The angles $(A_1,P_1,\frac{\pi}{2})$ and $(A_2,P_2,-\frac{\pi}{2})$ are not equivalent.

Example 10-2c. Construct the angles in standard position denoted by the symbols $(0,X,\frac{\pi}{4})$, $(0,X,\frac{11\pi}{6})$, and $(0,X,-\frac{4\pi}{3})$. Construct two other angles which are co-terminal with each of these angles.

Solution: The solutions are shown in Figure 10-21. Recall that the length of the circumference of the unit circle is 2π.



$\theta = \frac{\pi}{4}$
O
X
$\theta = -\frac{7\pi}{4}$
O
X
x
$\theta = \frac{9\pi}{4}$
O
x
$\theta = \frac{11\pi}{6}$
O
X
x
$\theta = -\frac{\pi}{6}$
O
X
x
$\theta = -\frac{13\pi}{6}$
O
X
x
$\theta = -\frac{4\pi}{3}$
O
X
x
$\theta = \frac{2\pi}{3}$
O
X
x
$\theta = \frac{8\pi}{3}$
O
X
x

Figure 10-21. The angles $(O,X,\frac{\pi}{4})$, $(O,X,\frac{11\pi}{6})$, $(O,X,-\frac{4\pi}{3})$, and two angles which are co-terminal with each.

The addition of paths suggests how angles are to be added. The following statements define the addition of two angles and the multiplication of an angle by a real number c:

$$(O,X,\theta_1) + (O,X,\theta_2) = (O,X,\theta_1 + \theta_2)$$

10-2b. $$c \cdot (O,X,\theta) = (O,X,c\theta).$$

The properties of these operations follow from the properties of the corresponding operations on the real numbers.

It is now clear that an angle (O,X,θ) in standard position is completely determined by the single real number θ. Henceforth, we shall speak of the angle θ and mean thereby the angle (O,X,θ). The sum of the angles θ_1 and θ_2 is $\theta_1 + \theta_2$; the addition of angles has all of the properties of the addition of real numbers. Furthermore, c times the angle θ is the angle $c\theta$; the multiplication of angles by a real number has all of the properties of the multiplication of real numbers.

Exercises 10-2

1. Given a unit circle with center A and a point P on it. Construct the angles.

   (a) (A,P,π)

   (b) $(A,P,-\frac{5\pi}{3})$

   (c) $(A,P,\frac{7\pi}{2})$

   (d) $(A,P,2)$

   (e) $(A,P,-4)$

   (f) $(A,P,-1.5)$.

2. Construct the following angles in standard position.

   (a) $(O,X,\frac{\pi}{3})$

   (b) (O,X,π)

   (c) $(O,X,-\frac{5\pi}{4})$

   (d) $(O,X,-\frac{7\pi}{3})$

   (e) $(O,X,-\frac{7\pi}{6})$

   (f) $(O,X\ \frac{3\pi}{2})$

3. Find two positive angles and two negative angles which are co-terminal with each of the angles in Exercise 2.

4. Construct the following angles in standard position and find one negative angle which is co-terminal with each one of them.

   (a) 0

   (b) $\frac{\pi}{6}$

   (c) $\frac{\pi}{4}$

   (d) $\frac{\pi}{3}$

   (e) $\frac{\pi}{2}$

   (f) $\frac{2\pi}{3}$

   (g) $\frac{3\pi}{4}$

   (h) $\frac{5\pi}{6}$

   (i) π

   (j) $\frac{7\pi}{6}$

   (k) $\frac{5\pi}{4}$

   (ℓ) $\frac{4\pi}{3}$

   (m) $\frac{3\pi}{2}$

   (n) $\frac{5\pi}{3}$

   (o) $\frac{7\pi}{4}$

   (p) $\frac{11\pi}{3}$

   (q) 2π

   (r) $\frac{16\pi}{9}$

10-3. Radian Measure.

We have defined the signed angle (A,P,θ) in terms of the path (P,θ) on the unit circle whose center is A. The real number θ is called the radian measure of the angle (A,P,θ). It follows from the definition of equivalent angles given in Section 10-2 that any two equivalent angles (A_1,P_1,θ_1) and (A_2,P_2,θ_2) have the same radian measure θ, where $\theta = \theta_1 = \theta_2$. The statement "the angle θ" usually means "the signed angle in standard position whose radian measure is θ". Of course there are infinitely many other angles that have the same radian measure θ.

The radian measure of angles is especially useful because there exists a simple relation between the length of an arc of a circle and the radian measure of the angle subtended at the center of the circle. Figure 10-3a shows an arc $P'Q'$ of length s on a circle of radius r, and the corresponding arc PQ of length θ on a circle of radius 1. By a theorem on similar sectors of circles, we have

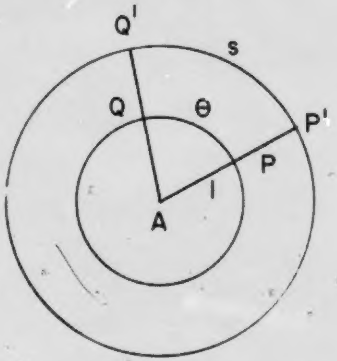


Figure 10-3a.
Arc s on a circle of radius r.

$$\frac{\text{arc } PQ}{1} = \frac{\text{arc } P'Q'}{r} \quad \text{or} \quad \theta = \frac{s}{r}.$$

But θ is the radian measure of the angle formed by the rays $\overrightarrow{AP'}$ and $\overrightarrow{AQ'}$. Thus, the formula

$$\theta = \frac{s}{r} \qquad \text{10-3a}$$

gives the radian measure of the angle in terms of radius of the circle and the length of the intercepted arc. Formula 10-3a can be stated also in the form

$$s = r\theta. \qquad \text{10-3b}$$

Formula 10-3b gives the length of an arc in terms of the radius of a circle and the radian measure of the subtended angle.

Example 10-3a. Find the radian measure of the angle subtended at the center of a circle of radius r by one-fourth of the circumference.

Solution: The length of the circumference of a circle of radius r is $2\pi r$, and one-fourth of the circumference is $\frac{\pi r}{2}$. By Equation 10-3a, $\theta = \frac{\frac{\pi r}{2}}{r} = \frac{\pi}{2}$.

Example 10-3b. An arc on a circle of radius 10 subtends an angle of 2.5 radians at the center. Find the length of the arc.

Solution: By Equation 10-3b, $s = 10 \times 2.5 = 25$. The reader should draw a figure.

Exercises 10-3

1. Compute the radian measures of the angles determined by the following values of s and r.

   (a) $s = 17$, $r = 5$

   (b) $s = 10$, $r = 5$

   (c) $s = 8$, $r = 10$

   (d) $s = 4\pi$, $r = 4$

   (e) $s = 2$, $r = 5$

   (f) $s = 3\pi$, $r = 5$

   (g) $s = 6\pi$, $r = 10$

   (h) $s = \pi$, $r = 1$

2. Compute the lengths of the arcs determined by the following values of r and θ.

   (a) $r = 5$, $\theta = 0.2$

   (b) $r = 5$, $\theta = \frac{\pi}{4}$

   (c) $r = 5$, $\theta = 2$

   (d) $r = 5$, $\theta = \frac{\pi}{6}$

   (e) $r = 10$, $\theta = 2.7$

   (f) $r = 10$, $\theta = \frac{\pi}{3}$

   (g) $r = 10$, $\theta = 3.2$

   (h) $r = 10$, $\theta = \frac{2\pi}{3}$

3. On a circle of radius 24 inches, find the length of an arc subtended by a central angle of:

   (a) $\frac{2}{3}$ radians . (c) 4 radians .

   (b) $\frac{3\pi}{5}$ radians .

4. Find the radius of a circle for which an arc of 15 inches long subtends an angle of:

   (a) 1 radian . (c) 3 radians .

   (b) $\frac{2}{3}$ radians .

10-4. Other Angle Measures.

The radian measure of angles was treated in the last Section. The size of an angle (A,P,θ) is determined by the length of the path (P,θ). In the radian system of measure, the unit of length used in measuring the length of the path is the length of the radius of the circle (see Equation 10-3a).

The circumference of the circle contains 2π of these units.

Another system of measure can be obtained by using the length of the circumference as the unit length for paths. The angle subtended by an arc one circumference in length is called one revolution. Since one circumference subtends an angle of 2π radians or 1 revolution, we have 1 revolution = 2π radians.

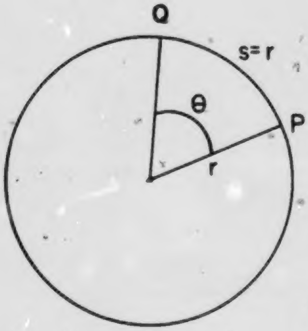


Figure 10-4a.
An angle θ of one radian.

A third system of measure results from using $\frac{1}{360}$ of the circumference of the circle as the unit of length. The angle subtended by $\frac{1}{360}$ of the circumference is called one degree. Since one circumference subtends an angle of 360 degrees or 1 revolution or 2π radians, we have the following basic statement of equivalents:

10-4a. $\quad$ 1 revolution = 2π radians = 360 degrees.

The degree is further subdivided into 60 equal parts called minutes (abbreviated min. and denoted by ', as in 10'); the minute finally is divided into 60 equal parts called seconds (abbreviated sec. and denoted by ", as in 20"). Thus,

10-4b $\quad$ $1^\circ = 60', \quad 1' = 60''$.

It is customary to measure angles in degrees, minutes, and seconds in surveying and in the solution of triangles. The radian, however, is the simplest unit for measuring angles in those problems which involve the differential and integral calculus.

Example 10-4a. Find the measure of each of the following angles in the other two systems: $\frac{\pi}{6}$ radians, $\frac{3}{2}$ rev., 150°.

Solution: From Equation 10-4a,

$$\pi \text{ radians} = 180^\circ, \qquad \pi = \frac{1}{2} \text{ rev.}$$

or

$$\frac{\pi}{6} \text{ radians} = 30^\circ, \qquad \frac{\pi}{6} = \frac{1}{12} \text{ rev.}$$

Similarly,

$$1 \text{ rev.} = 2\pi \text{ radians}, \qquad 1 \text{ rev.} = 360^\circ$$

$$\frac{3}{2} \text{ rev.} = 3\pi \text{ radians}, \qquad \frac{3}{2} \text{ rev.} = 540^\circ;$$

and

$$1^\circ = \frac{1}{360} \text{ rev.}, \qquad 1^\circ = \frac{\pi}{180} \text{ radians}$$

$$150^\circ = \frac{5}{12} \text{ rev.}, \qquad 150^\circ = \frac{5\pi}{6} \text{ radians} .$$

Exercises 10-4

1. Express the following in degrees.

(a) 3 revolutions

(b) $-\frac{3}{4}$ revolution

(c) $\frac{5}{8}$ revolution

(d) $-\frac{5}{6}$ revolution

(e) .125 revolution

(f) .833 revolution

(g) -1.5 revolutions

(h) $-2\frac{1}{2}$ revolutions

2. Express the following in revolutions.

(a) 135°

(b) -60°

(c) 210°

(d) -150°

(e) $67^\circ 30'$

(f) 930°

(g) -485°

(h) 360°

3. Express the angle as a multiple of π radians.

(a) 30°

(b) -25°

(c) -160°

(d) 135°

(e) 36°

(f) $75^\circ 30'$

(g) $-112^\circ 40'$

(h) -315°

(i) -180°

(j) 300°

(k) -90°

(ℓ) 880°

4. Express the following in degrees.

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\frac{-5\pi}{6}$

(d) $\frac{7\pi}{10}$

(e) $\frac{-5\pi}{12}$

(f) $\frac{7\pi}{15}$

(g) $\frac{-4\pi}{3}$

(h) $\frac{\pi}{15}$

(i) $\frac{-17\pi}{15}$

(j) 3π

(k) 2

(ℓ) -3.6

5. In a triangle, one angle is 36° and another is $\frac{2}{3}\pi$ radians. Find the third angle in radians.

6. Through how many radians does the minute hand of a clock revolve in 40 minutes?

10-5. Definitions of the Trigonometric Functions.

We shall define the trigonometric functions in this section. Some functions, such as the logarithmic and exponential functions, have names; the trigonometric functions also have names. The situation is sometimes confusing because several different but closely related functions have been given the same name. First we shall define the trigonometric functions of angles in standard position. Next, we shall define the trigonometric functions of arbitrary angles, and finally we shall define certain additional trigonometric functions which are closely related to the trigonometric functions of angles.

Definition 10-5a. Let $(0,X,\theta)$ be any angle in standard position, and let (x_0,y_0) be the intersection of its terminal side with the standard unit circle. Then

sine of $(0,X,\theta) = y_0$ $\qquad$ $\sin\theta = y_0$

cosine of $(0,X,\theta) = x_0$ $\qquad$ $\cos\theta = x_0$

tangent of $(0,X,\theta) = \frac{y_0}{x_0}$ $\qquad$ $\tan\theta = \frac{y_0}{x_0}$ provided $x_0 \neq 0$

cotangent of $(0,X,\theta) = \frac{x_0}{y_0}$ $\qquad$ $\cot\theta = \frac{x_0}{y_0}$ provided $y_0 \neq 0$

secant of $(0,X,\theta) = \frac{1}{x_0}$ $\qquad$ $\sec\theta = \frac{1}{x_0}$ provided $x_0 \neq 0$

cosecant of $(0,X,\theta) = \frac{1}{y_0}$ $\qquad$ $\csc\theta = \frac{1}{y_0}$ provided $y_0 \neq 0$.

where the statements on the right are abbreviations for the statements on the left.

These definitions do not enable us to calculate these six functions except in a few special cases since it is usually not possible to find the coordinates of a point on the terminal side of the angle (O,X,θ). In certain important special cases, however, the calculation is possible as shown in the following examples.

Example 10-5a. Find all six trigonometric functions of 30°.

Solution: Figure 10-5a shows the angle $(O,X,30^\circ)$. The terminal side of this angle intersects the standard unit circle in the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. Then

$$\sin 30^\circ = \frac{1}{2} \qquad \cot 30^\circ = \sqrt{3}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} \qquad \sec 30^\circ = \frac{2\sqrt{3}}{3}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \csc 30^\circ = 2 .$$

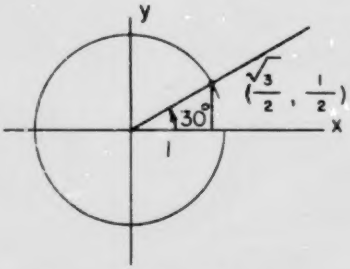


Figure 10-5a.
The angle $(O,X,30^\circ)$.

Example 10-5b. Find all six trigonometric functions of 120°.

Solution: Figure 10-5b shows the angle $(O,X,120^\circ)$. The terminal side of this angle intersects the standard unit circle in the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then

$$\sin 120^\circ = \frac{\sqrt{3}}{2} \qquad \cot 120^\circ = -\frac{\sqrt{3}}{3}$$

$$\cos 120^\circ = -\frac{1}{2} \qquad \sec 120^\circ = -2$$

$$\tan 120^\circ = -\sqrt{3} \qquad \csc 120^\circ = \frac{2\sqrt{3}}{3}.$$

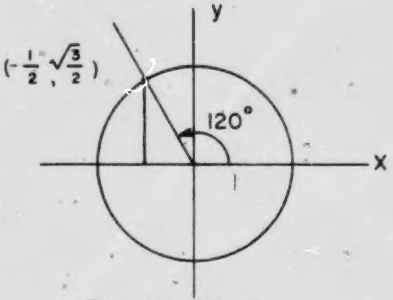


Figure 10-5b.
The angle $(O,X,120^\circ)$.

Example 10-5c. Find all six trigonometric functions of 270°.

Solution: Figure 10-5c shows the angle $(0,X,270^\circ)$. The terminal side of this angle intersects the standard unit circle in $(0,-1)$. Tan 270° and sec 270° are not defined since $x = 0$. The other values are

$\sin 270^\circ = -1$ $\qquad$ $\cot 270^\circ = 0$

$\cos 270^\circ = 0$ $\qquad$ $\csc 270^\circ = -1$.

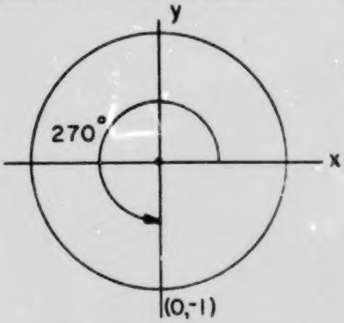


Figure 10-5c.
The angle $(0,X,270^\circ)$.

Definition 10-5b. Let (A,P,θ) be any angle, and let $(0,X,\theta)$ be the unique angle in standard position to which it is equivalent. Then

$$\sin (A,P,\theta) = \sin (0,X,\theta) \qquad \cot (A,P,\theta) = \cot (0,X,\theta)$$
$$\cos (A,P,\theta) = \cos (0,X,\theta) \qquad \sec (A,P,\theta) = \sec (0,X,\theta)$$
$$\tan (A,P,\theta) = \tan (0,X,\theta) \qquad \csc (A,P,\theta) = \csc (0,X,\theta).$$

If we pair with each signed angle (A,P,θ) the real number $\sin (A,P,\theta)$, we define a function whose domain is the set of all signed angles. It follows from Definition 10-5a that its range is $\{x: -1 \leq x \leq 1\}$. This function is denoted by $\sin \theta$, and θ is most commonly measured in degrees. Pairing $\cos (A,P,\theta)$ with (A,P,θ) defines a function whose domain is the set of all signed angles and whose range is $\{x: -1 \leq x \leq 1\}$; it is denoted by $\cos \theta$. The functions $\tan \theta$, $\cot \theta$, $\sec \theta$, and $\csc \theta$ are defined in a similar manner. The functions $\sin \theta$, ---, $\csc \theta$ are called the six trigonometric functions.

Theorem 10-5a. Let θ be any angle in standard position whose terminal side does not lie along one of the axes, and let $P(x,y)$ be any point on its terminal side. Then

$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$ $\qquad$ $\cot\theta = \frac{x}{y}$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ $\qquad$ $\sec\theta = \frac{\sqrt{x^2+y^2}}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad$ $\csc\theta = \frac{\sqrt{x^2+y^2}}{y}$

Proof: Let r be the distance from O to P. Then $r = \sqrt{x^2+y^2}$. The equation of the line through O and P is

$$y = \frac{y_0}{x_0}x,$$

where (x_0,y_0) is the point at which this line intersects the standard unit circle.

The point $P(x,y)$ is one of the intersections of this line with the circle $x^2+y^2=r^2$. The two intersections are found by solving the following system:

$$x^2+y^2=r^2$$
$$y = \frac{y_0}{x_0}x.$$

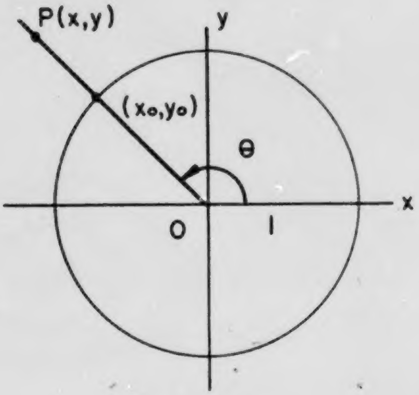


Figure 10-5d.
The figure for
Theorem 10-5a.

The two solutions are (rx_0,ry_0) and $(-rx_0,-ry_0)$ (remember that $x_0^2+y_0^2=1$). Since a ray lies entirely in one quadrant, the signs of the coordinates of P are the same as the signs of the coordinates of (x_0,y_0). It follows that the coordinates of P are (rx_0,ry_0). Thus

$$x = rx_0, \quad y = ry_0$$

$$x_0 = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$y_0 = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}.$$

Then, by Definition 10-5a,

$$\sin \theta = y_0 = \frac{y}{\sqrt{x^2 + y^2}},$$

and the other statements in the conclusion of the theorem are obtained in the same way.

Remark: In the course of the proof of the last theorem, we proved the following important fact: If (x_0,y_0) is one point on the terminal side of an angle, then all points on this terminal side have coordinates of the form

10-5a $(rx_0,ry_0) \quad r > 0.$

We shall now define a second set of trigonometric functions. This second set is highly important in more advanced mathematics and also in this course. This second set of functions is so closely related to the first set that the two are often confused.

Let (C,X,θ) be an angle in standard position, and let θ be its radian measure. If we pair with the real number θ, the real number $\sin (C,X,\theta)$ as defined in Definition 10-5a, we define a function whose domain is the set of all real numbers and whose range is $\{x: -1 \leq x \leq 1\}$. This function is obviously quite distinct from the function defined in Definition 10-5b. The domain of the former function is the set of all signed angles, but the domain of the present function is the set of all real numbers. It would be appropriate to denote the former function by $\sin (A,P,\theta)$ and the latter function by $\sin \theta$. Unfortunately, both are denoted by $\sin \theta$, but it will usually be clear from the context which is intended. It will usually be true that $\sin 60^\circ$ means the sine of the angle whose measure is 60°, whereas,

$\sin \frac{\pi}{3}$ means the sign of the number $\frac{\pi}{3}$.

In the same way, we define the functions $\cos \Theta$, $\tan \Theta$, $\cot \Theta$, $\sec \Theta$, and $\csc \Theta$, whose domains are the set of all real numbers. If Θ is the measure of (O,X,Θ) in degrees, and if we associate with the real number Θ, the real number $\sin(O,X,\Theta)$, we have another function whose domain is the set of all real numbers. This function is closely related to the one already defined, and it will not be considered further in this course.

The definitions of the six trigonometric functions can be stated in a special way that is highly useful for an acute angle in a right triangle. Let A,B,C be a right triangle as shown in Figure 10-5e, and let α denote the angle at the vertex A.

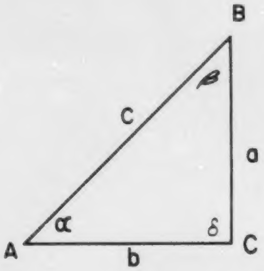


Figure 10-5e.
Functions of an acute angle.

Theorem 10-5b. If α is the angle at the vertex A of the right triangle shown in Figure 10-5e, then

$$\sin \alpha = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos \alpha = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan \alpha = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\cot \alpha = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}$$

$$\sec \alpha = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\csc \alpha = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

Proof: In order to find the trigonometric functions of α, we must first take an equivalent angle in standard position. Figure 10-5f shows such an angle. The point $P(b,a)$ is one point on the terminal side of this angle. The statements in the conclusion of the theorem now follow from Definition 10-5b and Theorem 10-5a (remember that $c = \sqrt{a^2 + b^2}$).

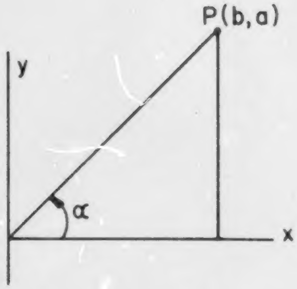


Figure 10-5f.
An angle in standard position equivalent to $\angle A$.

Exercises 10-5

1. Sketch the angle θ in the standard position and find $\sin \theta$, $\cos \theta$ and $\tan \theta$ when the following points are on the terminal side of the angle θ.
   - (a) $P(-4,3)$
   - (b) $P(5,-12)$
   - (c) $P(-1,-1)$
   - (d) $P(2,3)$
   - (e) $P(-2,4)$
   - (f) $P(-7,-24)$
   - (g) $P(3,-5)$
   - (h) $P(4,1)$

2. In each of the following sketch the angle θ and find the other five functions of θ.
   - (a) $\tan \theta = \frac{3}{4}$, θ in quadrant I
   - (b) $\cos \theta = \frac{1}{2}$, θ in quadrant IV
   - (c) $\sin \theta = -\frac{2}{5}$, θ in quadrant IV
   - (d) $\tan \theta = -\frac{5}{3}$, θ in quadrant II
   - (e) $\cos \theta = -\frac{4}{7}$, θ in quadrant III
   - (f) $\sin \theta = \frac{3}{8}$, θ in quadrant II

3. Let P be the set $\{a, b, c\}$ and let Q be the set $\{\sin\alpha, \cos\alpha, \tan\alpha, \tan\beta, \cos\beta, \sin\beta\}$. It can be proved that if any 2 members of set P are given, then all the other members of P and Q can be expressed in terms of these given members.

(a) If $a = 3$
$b = 4$
$\tan\beta = ?$
$\cos\alpha = ?$
$c = ?$

(b) If $b = 12$
$c = 13$
$a = ?$
$\sin\beta = ?$
$\tan\alpha = ?$

(c) If $a = 5$
$c = 11$
$b = ?$
$\cos\alpha = ?$
$\tan\beta = ?$
$\sin\alpha = ?$

(d) If $a = 12$
$b = 7$
$c = ?$
$\sin\alpha = ?$
$\cos\beta = ?$
$\tan\alpha = ?$

4. Let P and Q be the sets given in Problem 4. It can be proved that if a member of set P is given and if a member of set Q is given, then all other members of P and Q can be expressed in terms of the given members.

(a) If $a = 12$
$\cos\alpha = \frac{3}{5}$
$b = ?$
$c = ?$
$\tan\beta = ?$
$\sin\alpha = ?$

(b) $b = 15$
$\sin\alpha = \frac{2}{3}$
$c = ?$
$a = ?$
$\cos\beta = ?$
$\tan\alpha = ?$

(c) $c = 20$
$\tan\beta = 2$
$a = ?$
$b = ?$
$\cos\alpha = ?$
$\sin\alpha = ?$

(d) If $c = 8$
$\sin\alpha = \frac{5}{6}$
$b = ?$
$a = ?$
$\cos\beta = ?$
$\tan\alpha = ?$

(e) $a = 2$
$\tan\alpha = 1.8$
$b = ?$
$c = ?$
$\sin\beta = ?$
$\cos\alpha = ?$

(f) $b = 10$
$\cos\beta = .8$
$a = ?$
$c = ?$
$\sin\alpha = ?$
$\tan\beta = ?$

10-6. Some Basic Properties of the Sine and Cosine.

In order to simplify the statements of some of the results in this section, it will be convenient to introduce the notion of primary angle.

Definition 10-6a. An angle $(0,X,\theta)$ in standard position is called a primary angle if and only if $0 \leq \theta < 360^\circ$ (or the equivalent condition in other units of measure).

Theorem 10-6a. Let (A,P,θ) be any angle. Then

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

Proof: Let $(0,X,\theta)$ be the unique equivalent angle in standard position. Let (x_0,y_0) be the point where the terminal side of $(0,X,\theta)$ intersects the standard unit circle. Then

$$x_0^2 + y_0^2 = 1$$

$$x_0 = \cos \theta, \quad y_0 = \sin \theta$$

$$(\sin \theta)^2 + (\cos \theta)^2 = 1,$$

and the proof is complete.

Theorem 10-6b. (Converse of Theorem 10-6a.) Let x_0 and y_0 be any two numbers such that $x_0^2 + y_0^2 = 1$. Then there is one and only one primary angle $(0,X,\theta)$ such that $\cos \theta = x_0$, $\sin \theta = y_0$.

Proof: The point $P(x_0,y_0)$ is on the standard unit circle. Let $(0,X,\theta)$ be the primary angle whose terminal side passes through the point $P(x_0,y_0)$. Then $\cos \theta = x_0$ and $\sin \theta = y_0$. If $(0,X,\theta')$ is any other primary angle, then its terminal side does not pass through $P(x_0,y_0)$. Thus, it is not true that $\cos \theta' = x_0$ and $\sin \theta' = y_0$. The proof of the theorem is complete.

Theorem 10-6a emphasizes the following corollary, which has already been observed from the definitions in Section 10-5.

Corollary 10-6a. For all angles (A,P,θ)

$$-1 \leq \sin \theta \leq 1$$

$$-1 \leq \cos \theta \leq 1.$$

Corollary 10-6b. If y_0 is any number such that $-1 < y_0 < 1$, there are exactly two primary angles (O,X,θ) such that $\sin \theta = y_0$. These angles have $\sqrt{1 - y_0^2}$ and $-\sqrt{1 - y_0^2}$ for their respective cosines.

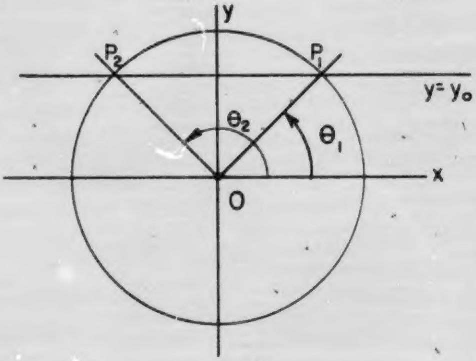


Figure 10-6a.
Two angles for which $\sin \theta = y_0$.

Proof: The line $y = y_0$ intersects the standard unit circle in the two distinct points

$P_1(\sqrt{1 - y_0^2}, y_0)$

$P_2(-\sqrt{1 - y_0^2}, y_0)$.

There are two primary angles θ_1 and θ_2 whose terminal sides pass through P_1 and P_2. Then

$$\cos \theta_1 = \sqrt{1 - y_0^2} \text{ and } \cos \theta_2 = -\sqrt{1 - y_0^2}.$$

Corollary 10-6c. If x_0 is any number such that $-1 < x_0 < 1$, there are exactly two primary angles (O,X,θ) such that $\cos \theta = x_0$. The sines of these angles are $\sqrt{1 - x_0^2}$ and $-\sqrt{1 - x_0^2}$ respectively.

Proof: The line $x = x_0$ intersects the circle $x^2 + y^2 = 1$ in the two distinct points

$$P_1(x_0, \sqrt{1 - x_0^2}), \quad P_2(x_0, -\sqrt{1 - x_0^2}).$$

There are two primary angles (O,X,θ_1) and (O,X,θ_2) whose terminal sides pass through P_1 and P_2. Then

$\sin \theta_1 = \sqrt{1 - x_0^2}$

$\sin \theta_2 = -\sqrt{1 - x_0^2}.$

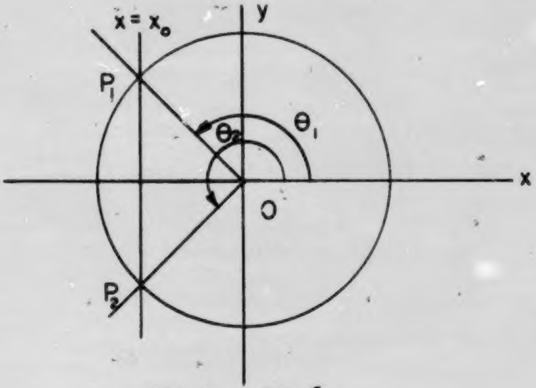


Figure 10-6b.
Two angles for
which $\cos \theta = x_0$.

Corollary 10-6d. There is exactly one primary angle whose sine is 1, and it is 90°; there is exactly one primary angle whose cosine is 1, and it is 0°. There is exactly one primary angle whose sine is -1, and it is 270°; there is exactly one primary angle whose cosine is -1, and it is 180°.

Let θ and $\theta + n \cdot 360^\circ$, where n is an integer, be two angles in standard position. These two angles have the same terminal side (they are called co-terminal angles), and hence, the six trigonometric functions of $\theta + n \cdot 360^\circ$ are equal respectively to the six trigonometric functions of θ. Hence, if

$$\sin \theta = y_0, \quad \cos \theta = x_0,$$

then

$$\sin (\theta + n \cdot 360^\circ) = y_0, \quad \cos (\theta + n \cdot 360^\circ) = x_0$$

for

$$n = 0, \pm 1, \pm 2, \ \ldots .$$

Exercises 10-6

1. Sketch all the angles between 0° and 360° in standard position which satisfy the following conditions and give the values of the other functions for these angles.

   (a) $\sin\theta = \frac{24}{25}$

   (b) $\cos\theta = -\frac{4}{5}$

   (c) $\tan\theta = -2$

   (d) $\sin\theta = \frac{\sqrt{3}}{\sqrt{5}}$

   (e) $\tan\theta = \frac{7}{24}$

   (f) $\sin\theta = \frac{\sqrt{11}}{6}$

   (g) $\cos\theta = \frac{-5}{\sqrt{34}}$

   (h) $\sin\theta = \frac{\sqrt{5}}{3}$

2. In what quadrant will the terminal side of θ lie if:

   (a) $\sin\theta$ and $\cos\theta$ are both positive?

   (b) $\tan\theta$ is positive and $\cos\theta$ is negative?

   (c) $\sin\theta$ is positive and $\tan\theta$ is negative?

   (d) $\cos\theta$ and $\tan\theta$ are both negative?

   (e) $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all negative?

3. Find the value of $\cos^2\theta - \sin^2\theta$ when $\tan\theta = -\frac{3}{4}$ and $\cos\theta$ is negative.

4. Find the value of $\frac{2\tan\theta}{1-\tan^2\theta}$ when $\cos\theta = -\frac{3}{7}$ and $\tan\theta$ is positive.

5. Prove the relation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ provided $\theta \neq (2k+1)90^\circ$ where k is an integer.

6. Use the relation $\sin^2\theta + \cos^2\theta = 1$ to prove:

   (a) $1 + \tan^2\theta = \sec^2\theta \quad [\theta \neq (2k+1)90^\circ]$.

   (b) $1 + \cot^2\theta = \csc^2\theta \quad [\theta \neq k\cdot 180^\circ]$.

7. Prove that the range of the tangent function is the set of all real numbers.

10-7. Trigonometric Functions of Special Angles.

The values of the trigonometric functions can be obtained by simple geometrical considerations for certain special angles. These are the angles for which the coordinates of the point (x_0,y_0), where the terminal side intersects the unit circle, can be computed. We list these angles θ in a table which shows the degree measure of θ, the radian measure of θ, the coordinates (x_0,y_0), and the values of the six trigonometric functions.

Table 10-7a. Trigonometric functions of special angles.

Degree Measure of θ	Radian Measure of θ	(X_0, Y_0)	cos θ	sin θ	Tan θ	Sec θ	Csc θ	Cot θ
0	0	$(1, 0)$	1	0	0	1	Undefined	Undefined
30	$\frac{\pi}{6}$	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2	$\sqrt{3}$
45	$\frac{\pi}{4}$	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	$(0, 1)$	0	1	Undefined	Undefined	1	0
120	$\frac{2\pi}{3}$	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	-2	$\frac{2}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$
135	$\frac{3\pi}{4}$	$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\frac{2}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$	-1
150	$\frac{5\pi}{6}$	$(-\frac{\sqrt{3}}{2}, \frac{1}{2})$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$	2	$-\sqrt{3}$
180	π	$(-1, 0)$	-1	0	0	-1	Undefined	Undefined
210	$\frac{7\pi}{6}$	$(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$	-2	$\sqrt{3}$
225	$\frac{5\pi}{4}$	$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\frac{2}{\sqrt{2}}$	$-\frac{2}{\sqrt{2}}$	1
240	$\frac{4\pi}{3}$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$	-2	$-\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
270	$\frac{3\pi}{2}$	$(0, -1)$	0	-1	Undefined	Undefined	-1	0
300	$\frac{5\pi}{3}$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	2	$-\frac{2}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$
315	$\frac{7\pi}{4}$	$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\frac{2}{\sqrt{2}}$	$-\frac{2}{\sqrt{2}}$	-1
330	$\frac{11\pi}{6}$	$(\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	-2	$-\sqrt{3}$

It is not necessary to memorize the results in Table 10-7a, but it is important to learn the methods by which these results are obtained.

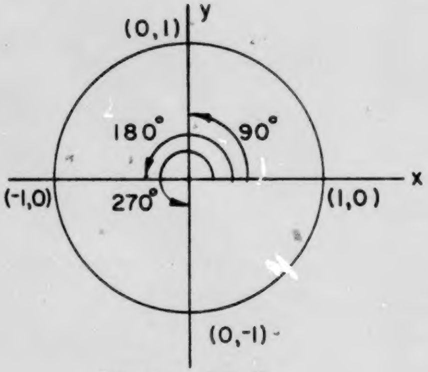


Figure 10-7a.
The special angles 0°, 90°, 180°, and 270°.

Consider first the angles 0°, 90°, 180°, and 270°. Figure 10-7a shows a point (x_0,y_0) on the terminal side of each of these angles. The entries in Table 10-7a for these angles are obtained by applying the definitions in Section 10-5. Observe that certain of the trigonometric functions of these angles are undefined.

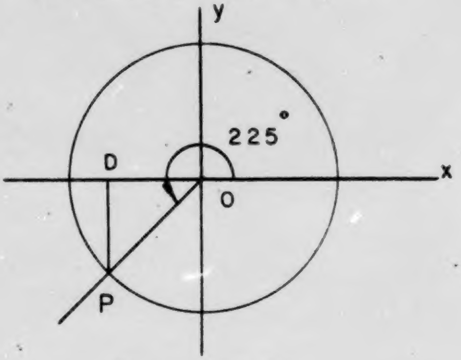


Figure 10-7b.
The special angle 225°.

The angle 225°, shown in Figure 10-7b, will be used to illustrate the method of finding the trigonometric functions of the special angles 45°, 135°, 225°, and 315°. The triangle OPD is an isosceles right triangle. Since the length $|OP|$ of its hypotenuse is 1, we find $|OD| = |DP| = \frac{\sqrt{2}}{2}$, and the coordinates of P are $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. An application of the definitions leads to the results given in the table. The trigonometric functions of 45°, 135°, and 315° can be obtained in a similar manner.

The angle 120°, shown in Figure 10-7c, will be used to illustrate the method of finding the trigonometric functions of the special angles 60°, 120°, 240°, and 300°. The triangle OPD is a right triangle whose acute angles are 30° and 60°. Since $|OP| = 1$, we find $|OD| = \frac{1}{2}|OP| = \frac{1}{2}$. Then $|DP| = \frac{\sqrt{3}}{2}$, and the coordinates of P are $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. An application of the definitions leads to the results given in the table. The trigonometric functions of 60°, 240°, and 300° can be obtained in a similar manner.

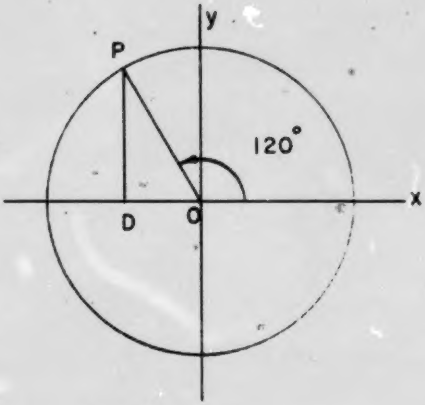


Figure 10-7c.
The special angle 120°.

The angle 330°, shown in Figure 10-7d, will be used to illustrate the method of finding the trigonometric functions of 30°, 150°, 210°, and 330°. The triangle OPD is again a triangle whose acute angles are 30° and 60°. Then $|OD| = \frac{\sqrt{3}}{2}$ and $|DP| = \frac{1}{2}$, and the coordinates of P are $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. An application of the definitions leads to the result given in Table 10-7a. The trigonometric functions of 30°, 150°, and 210° can be obtained in a similar manner.

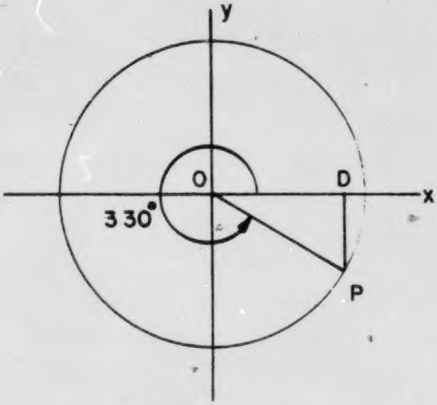


Figure 10-7d.
The special angle 330°.

Exercises 10-7

1. Evaluate the following:

   (a) $\sin 150^\circ \ \tan 210^\circ - \cos 135^\circ \ \sin 240^\circ$.

   (b) $\cos 90^\circ - \sin 300^\circ + \tan 225^\circ - \cos 150^\circ$.

   (c) $\sin 270^\circ + \tan 180^\circ \ \cos 90^\circ$.

2. Find all the functions of the following angles without using a table.

   (a) 210°
   (b) -135°
   (c) 315°
   (d) -225°.

3. Show that $\cos^2 \theta + \sin^2 \theta = 1$, for:

   (a) $\theta = 45^\circ$
   (b) $\theta = 150^\circ$
   (c) $\theta = 330^\circ$
   (d) $\theta = \frac{\pi}{6}$ radians
   (e) $\theta = \frac{7\pi}{4}$ radians
   (f) $\theta = \frac{2\pi}{3}$ radians.

4. Show that:

   (a) $\sin (60^\circ + 60^\circ) \neq \sin 60^\circ + \sin 60^\circ$

   (b) $\cos (90^\circ + 60^\circ) \neq \cos 90^\circ + \cos 60^\circ$

   (c) $\sin (180^\circ + 60^\circ) \neq \sin 180^\circ + \sin 60^\circ$

   (d) $\cos (150^\circ - 60^\circ) \neq \cos 150^\circ - \cos 60^\circ$

   (e) $\sin (300^\circ - 120^\circ) \neq \sin 300^\circ - \sin 120^\circ$.

5. Verify the following:

   (a) $1 - \cos^2 60^\circ = \sin^2 60^\circ$

   (b) $\sin 60^\circ \ \cos 30^\circ + \cos 60^\circ \ \sin 30^\circ = 1$

   (c) $\cos 60^\circ \ \cos 30^\circ - \sin 60^\circ \ \sin 30^\circ = 0$

   (d) $\cos 30^\circ = \sqrt{\frac{1 + \cos 60^\circ}{2}}$

   (e) $\sin 30^\circ = \sqrt{\frac{1 - \cos 60^\circ}{2}}$

   (f) $2 \sin 45^\circ \ \cos 45^\circ = 1$.

6. Which of the following statements are correct? Justify your answer.

   (a) $\sin \theta = 3$

   (b) $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$

   (c) $\sin 30^\circ + \sin 60^\circ = \sin 90^\circ$

   (d) $\cos^2 45^\circ + \sin^2 45^\circ = \sin 90^\circ$

   (e) $\cos 45^\circ = \frac{1}{2} \cos 90^\circ$

   (f) $\sin 45^\circ \ \cos 45^\circ = \frac{1}{2} \sin 90^\circ$

   (g) $\sin 30^\circ = \frac{1}{2} \sin 90^\circ$

   (h) $\sin^2 30^\circ + \cos^2 330^\circ = 1$

   (i) $\tan 45^\circ = \dfrac{\sin 45^\circ}{\cos 315^\circ}$

   (j) $\cos 30^\circ + 2 \cos 60^\circ = \cos 150^\circ$

10-8. Tables of Trigonometric Functions.

In Section 10-7, we explained how to find $\sin \theta$, ---, $\csc \theta$ for certain special values of θ. There is no elementary method for computing the six trigonometric functions of an arbitrary angle θ. In a typical case, the six trigonometric functions of θ are irrational numbers which would be represented by non-terminating decimals. These values can be calculated to any desired degree of accuracy by methods developed in calculus. Tables of the trigonometric functions are available. Table 10-8a gives sines, cosines, and tangents for the angles 1°, 2°, ..., 90°.

We shall now describe certain characteristics of trigonometric tables.

(1) Since the values of the trigonometric functions are usually irrational numbers, and since the tables give these values only to three (or four, or five) decimal places, the values in the tables are usually not exact. Table 10-8a is correct to three

decimal places, but tables correct to five, seven, or more decimal places are available for calculations which require greater accuracy.

(2) Table 10-8a gives sines, cosines, and tangents of the angles 0°, 1°, ..., 90°, but it does not give these functions for other angles such as 37.8°. It will be shown that the approximate values of the functions of these angles can be obtained by interpolation.

(3) Table 10-8a does not contain any angles θ such that $\theta < 0^\circ$ or $\theta > 90^\circ$. We shall show that the approximate values of the sines, cosines, and tangents of all angles can be obtained from Table 10-8a.

First, we shall give some examples which involve interpolation. Linear interpolation has been explained already in Chapter 9 on logarithms and exponents, and the theory will not be repeated here.

Example 10-8a. Find $\cos 37.8^\circ$.

Solution: From Table 10-8a we find

$$\cos 37^\circ \approx .799$$

$$\cos 38^\circ \approx .788$$

$$\frac{x}{-11} = \frac{.8}{1} \quad \text{or} \quad x = -8.8$$

$$\cos 37.8^\circ \approx .790.$$

It is important to observe that $\cos \theta$ decreases as θ increases.

Table 10-8a.

Degrees	Radians	Sine	Cosine	Tangent
0	.000	0.000	1.000	0.000
1	.017	.018	1.000	.018
2	.035	.035	0.999	.035
3	.052	.052	.999	.052
4	.070	.070	.998	.070
5	.087	.087	.996	.088
6	.105	.105	.995	.105
7	.122	.122	.993	.123
8	.140	.139	.990	.141
9	.157	.156	.988	.158
10	.175	.174	.985	.176
11	.192	.191	.982	.194
12	.209	.208	.978	.213
13	.227	.225	.974	.231
14	.244	.242	.970	.249
15	.262	.259	.966	.268
16	.279	.276	.961	.287
17	.297	.292	.956	.306
18	.314	.309	.951	.325
19	.332	.326	.946	.344
20	.349	.342	.940	.364
21	.367	.358	.934	.384
22	.384	.375	.927	.404
23	.401	.391	.921	.425
24	.419	.407	.914	.445
25	.436	.423	.906	.466
26	.454	.438	.899	.488
27	.471	.454	.891	.510
28	.489	.470	.883	.532
29	.506	.485	.875	.554
30	.524	.500	.866	.577
31	.541	.515	.857	.601
32	.559	.530	.848	.625
33	.576	.545	.839	.649
34	.593	.559	.829	.675
35	.611	.574	.819	.700
36	.628	.588	.809	.727
37	.646	.602	.799	.754
38	.663	.616	.788	.781
39	.681	.629	.777	.810
40	.698	.643	.766	.839
41	.716	.658	.755	.869
42	.733	.669	.743	.900
43	.751	.682	.731	.933
44	.768	.695	.719	.966
45	.785	.707	.707	1.000
46	0.803	0.719	0.695	1.036
47	.820	.731	.682	1.072
48	.838	.743	.669	1.111
49	.855	.755	.656	1.150
50	.873	.766	.643	1.192
51	.890	.777	.629	1.235
52	.908	.788	.616	1.280
53	.925	.799	.602	1.327
54	.942	.809	.588	1.376
55	.960	.819	.574	1.428
56	.977	.829	.559	1.483
57	.995	.839	.545	1.540
58	1.012	.848	.530	1.600
59	1.030	.857	.515	1.664
60	1.047	.866	.500	1.732
61	1.065	.875	.485	1.804
62	1.082	.883	.470	1.881
63	1.100	.891	.454	1.963
64	1.117	.899	.438	2.050
65	1.134	.906	.423	2.145
66	1.152	.914	.407	2.246
67	1.169	.921	.391	2.356
68	1.187	.927	.375	2.475
69	1.204	.934	.358	2.605
70	1.222	.940	.342	2.747
71	1.239	.946	.326	2.904
72	1.257	.951	.309	3.078
73	1.274	.956	.292	3.271
74	1.292	.961	.276	3.487
75	1.309	.966	.259	3.732
76	1.326	.970	.242	4.011
77	1.344	.974	.225	4.331
78	1.361	.978	.208	4.705
79	1.379	.982	.191	5.145
80	1.396	.985	.174	5.671
81	1.414	.988	.156	6.314
82	1.431	.990	.139	7.115
83	1.449	.993	.122	8.144
84	1.466	.995	.105	9.514
85	1.484	.996	.087	11.43
86	1.501	.998	.070	14.30
87	1.518	.999	.052	19.08
88	1.536	.999	.035	28.64
89	1.553	1.000	.018	57.29
90	1.571	1.000	.000	undefined

Example 10-8b. If $\sin \theta = .600$, what is θ?

Solution: From Table 10-8a we find

$$\sin 36^\circ \approx .588$$
$$\sin 37^\circ \approx .602$$
$$\frac{x}{1} = \frac{.12}{.14} \quad \text{or} \quad x = .9^\circ .$$

Thus, if $\sin \theta = .600$, then $\theta \approx 36.9^\circ$. From Corollary 10-6b we know that there is another primary angle θ such that $\sin \theta = .600$. A little later we shall show how to find this second primary angle. All other solutions of the equation $\sin \theta = .600$ can be obtained from the two primary solutions by the method explained at the end of Section 10-6.

We shall now explain how the trigonometric functions of any angle can be obtained from a table which gives the trigonometric functions of angles from 0° to 90°. We observe first that the functions of any angle are equal to the functions of a co-terminal angle which lies between 0° and 360°. For example, $\sin 473^\circ = \sin 113^\circ$. The problem is thus reduced to finding the functions of all angles between 0° and 360°. If θ is one of the special angles 0°, 90°, 180°, and 270°, it's functions can be obtained from Table 10-7a. For each angle θ, where $0^\circ \leq \theta < 360^\circ$ and θ is not one of the special angles, an angle θ_R called the reference angle of θ is defined by Table 10-8b.

Table 10-8b. The reference angle of θ

θ	The reference angle θ_R of θ
$0^\circ < \theta < 90^\circ$	$\theta_R = \theta$
$90^\circ < \theta < 180^\circ$	$\theta_R = 180^\circ - \theta$
$180^\circ < \theta < 270^\circ$	$\theta_R = \theta - 180^\circ$
$270^\circ < \theta < 360^\circ$	$\theta_R = 360^\circ - \theta$

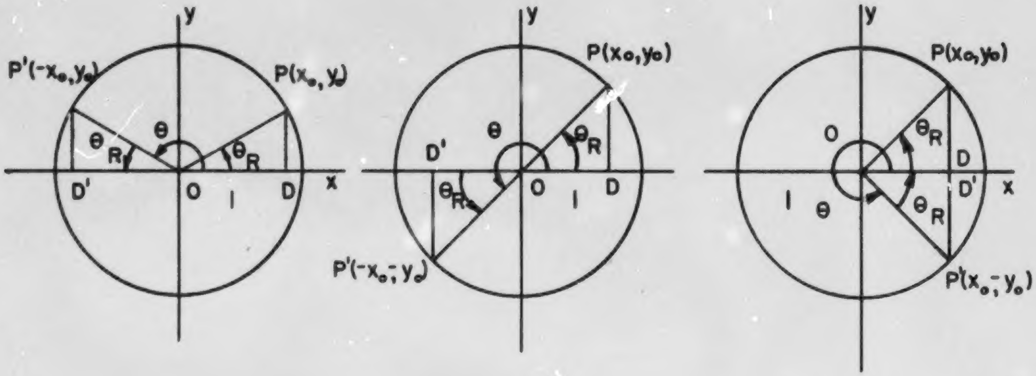


Figure 10-8a. Reference angles for the angle θ.

Figure 10-8a shows the reference angle θ_R for angles θ in quadrants II, III, and IV. The circles in this figure are the standard unit circles. Let $P(x_0,y_0)$ be the point where the terminal side of the reference angle θ_R intersects the unit circle, and let P' be the corresponding point on the terminal side of θ. The triangle OP'D' is congruent to the triangle OPD in every case. Thus, the coordinates of P' are $(\pm x_0, \pm y_0)$.

Theorem 10-8a. Let θ be any angle such that $0^\circ \leq \theta < 360^\circ$ and such that θ is not an integral multiple of 90°, and let θ_R be the reference angle of θ. Then

$$\sin\theta = \pm\sin\theta_R \qquad \cot\theta = \pm\cot\theta_R$$
$$\cos\theta = \pm\cos\theta_R \qquad \sec\theta = \pm\sec\theta_R$$
$$\tan\theta = \pm\tan\theta_R \qquad \csc\theta = \pm\csc\theta_R$$

Proof: Examine Figure 10-8a. For every angle θ of the kind specified in the theorem, $\sin \theta = y_0$ or $\sin \theta = -y_0$. But $\sin \theta_R = y_0$. Thus, either $\sin \theta = \sin \theta_R$ or $\sin \theta = -\sin \theta_R$. The other statements in the conclusion of the theorem can be established in the same way.

Table 10-8c shows the signs of the six trigonometric functions for angles in the four quadrants. The results given in this table follow from the definitions in Section 10-5.

Table 10-8c. Signs of the Trigonometric Functions

Trigonometric Functions	Quadrants			
	I	II	III	IV
sin	+	+	-	-
cos	+	-	-	+
tan	+	-	+	-
cot	+	-	+	-
sec	+	-	-	+
csc	+	+	-	-

Theorem 10-8a and Table 10-8c enable us to find the six trigonometric functions of any angle from tables for all angles from 0° to 90°. The method will be explained by means of examples.

Example 10-8c. Find $\sin 603^\circ$ by using Table 10-8a.

Solution: The angles 603° and 243° are co-terminal; hence, $\sin 603^\circ = \sin 243^\circ$. By Table 10-8b, the reference angle of 243° is 63°. Thus, by Theorem 10-8a, $\sin 243^\circ = \pm \sin 63^\circ$. From Table 10-8a, $\sin 63^\circ \approx .891$. Since 243° is an angle in the third quadrant, $\sin 243^\circ$ is negative by Table 10-8c. Thus, $\sin 603^\circ = \sin 243^\circ = - \sin 63^\circ \approx - .891$. The entire solution, except for finding $\sin 63^\circ$ in the table of sines, should be geometrically obvious from Figure 10-8b.

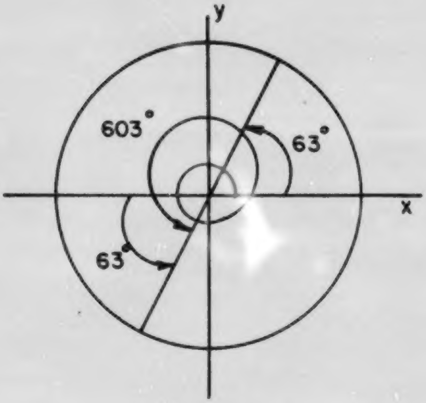


Figure 10-8b.
Graph of 603°
and its reference angle.

Example 10-8d. Find $\tan 328^\circ$.

Solution: The reference angle is 32°, and the tangent is negative in the fourth quadrant. Thus, $\tan 328^\circ = - \tan 32^\circ \approx - .625$. The reader should draw a figure.

Example 10-8e. Find $\cos \frac{5\pi}{6}$.

Solution: Since $\frac{5\pi}{6}$ is the radian measure of the angle, the reference angle is $\frac{\pi}{6}$ radians. Also, $\frac{5\pi}{6}$ is an angle in the second quadrant, where the cosine is negative. Thus, $\cos \frac{5\pi}{6} = - \cos \frac{\pi}{6} \approx - .866$. The reader should draw a figure.

Example 10-8f. Find $\sin 1046^\circ$.

Solution: Since $1046^\circ = 2(360^\circ) + 326^\circ$, the angles 1046° and 326° are co-terminal. Thus, $\sin 1046^\circ = \sin 326^\circ$. The reference angle of 326° is 34°, and $\sin 326^\circ$ is negative since the angle is in the fourth quadrant. Thus, $\sin 1046^\circ = \sin 326^\circ = - \sin 34^\circ \approx - .559$.

Example 10-8g. Find $\cos(-150^\circ)$.

Solution: The angles -150° and 210° are co-terminal; hence, $\cos(-150^\circ) = \cos 210^\circ$. The reference angle for 210° is 30°, and the cosine is negative in the third quadrant. Thus, $\cos(-150^\circ) = \cos 210^\circ = -\cos 30^\circ \approx -.866$.

With our tables available we are now equipped to discuss some examples of a simple and important application of the trigonometric functions — the indirect measurement of distances by triangulation.

Example 10-8h. At a point 439 feet from the base of a building the angle between the horizontal and the line to the top of the building (angle of elevation) is 31°. What is the height of the building?

Solution: In the right triangle ABC we have $\gamma = 90^\circ$ $\alpha = 31^\circ$ and $b = 439$ feet.

In this drawing we seek the height a of the building. According to the formula for the tangent of an acute angle of a right triangle we have

$$\tan 31^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{a}{439}$$

our Table 10-8a gives

$$\tan 31^\circ \approx .601 \quad .$$

Combining these two equations we have

$$\frac{a}{439} \approx .601 \ .$$

Therefore $a \approx 439(.601) \approx 264$, so that building is approximately 264 feet high.

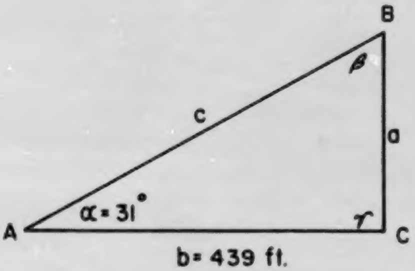


Figure 10-8c.

Example 10-8i. To measure the width of a river a stake was driven into the ground on the south bank directly south of a tree on the opposite bank. From a point 100 ft. due west of the stake, the tree was sighted and the angle between the line of sight and the east-west line measured. What is the width of the river if this angle was 60°?

Solution: The point from which the tree was sighted was taken due west of the stake so that the angle RST (Figure 10-8d) would be a right angle. From the formula for the tangent of an acute angle in a right triangle (Section 10-5) and Table 10-8a we have $\frac{r}{100} = \tan 60^\circ$ $= \sqrt{3}$ where r is the required width of the river.

$$y = 100\sqrt{3} \approx 173 \quad .$$

The river is approximately 173 feet wide.

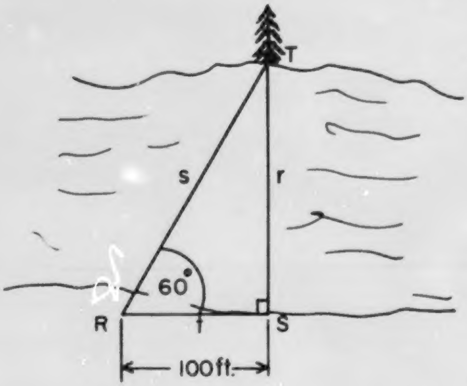


Figure 10-8d.

Example 10-8j. At the instant when the moon is exactly at half phase the angle between the line from the earth to the moon and the line from the earth to the sun is between 89° and 90°. Show that the distance from the earth to the sun is at least 50 times the distance from the earth to the moon.

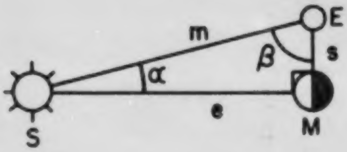


Figure 10-8e.

Solution: From Figure 10-8e we see that if the moon is exactly at half-phase the angle SME is a right angle. Since angle SEM $= \beta$ and $89^\circ < \beta < 90^\circ$, we have $0^\circ < \alpha < 1^\circ$. Then the distance m of the earth to the sun and the distance s from the earth to the moon are related thus

$$\sin \alpha = \frac{s}{m}$$

and from Table 10-8a

$$\sin \alpha < \sin 1^\circ \approx .018$$

so that

$$\frac{s}{m} < .018 = \frac{18}{1000} < \frac{20}{1000} = \frac{1}{50}$$

$$m > 50s .$$

Thus the distance from earth to the sun is at least 50 times the distance from earth to moon.

The essential step in these examples is the discovery and construction of a right triangle one of the sides of which is the length to be measured. In Sections 10 and 11 we will learn some further theorems about the trigonometric functions which will permit us to use more general triangles in a similar way. Before this we must discuss the trigonometric functions in more detail.

Exercises 10-8

1. What is the reference angle of each of the following?

   (a) 150°

   (b) 260°

   (c) $\frac{7\pi}{6}$ radians

   (d) 308°

   (e) 615°

   (f) -98°

   (g) -235°

   (h) $\frac{7\pi}{8}$ radians

   (i) $\frac{2\pi}{9}$ radians

   (j) $\frac{16\pi}{5}$ radians

2. Express the following in terms of the same function of the reference angle.

   (a) $\sin 165^\circ$

   (b) $\tan 190^\circ$

   (c) $\cos 212^\circ$

   (d) $\sin \frac{2\pi}{5}$

   (e) $\cos(-\frac{\pi}{3})$

   (f) $\tan(-\pi)$

   (g) $\sin 340^\circ$

   (h) $\sin 98^\circ$

   (i) $\tan 462^\circ$

   (j) $\cos(-160^\circ)$

   (k) $\sin(-195^\circ)$

   (ℓ) $\sin 305^\circ$

   (m) $\tan(-378^\circ)$

   (n) $\sin \frac{12\pi}{5}$

   (o) $\cos(-\frac{3\pi}{8})$

   (p) $\tan(-\frac{5\pi}{3})$

   (q) $\sin 335^\circ$

   (r) $\cos(-3\pi)$

   (s) $\sin 600^\circ$

   (t) $\tan \frac{33\pi}{4}$

3. A wire 35 feet long is stretched from level ground to the top of a pole 25 feet high. Find the angle between the pole and the wire.

4. A kite string 200 yards long makes an angle of 70° with the ground. How high is the kite?

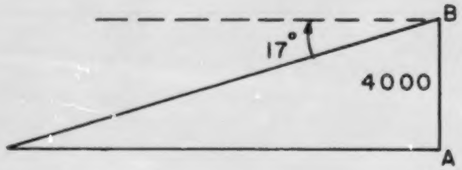
5. From the top of a rock which rises vertically 326 feet out of the water, the angle between the line of sight of a boat and the horizontal (angle of depression) is 24°. Find the distance of the boat from the base of the rock.

6. The edge of the Great Pyramid is 609 feet and makes an angle of 52° with the horizontal plane. What is the height of the pyramid?

7. A gun G shoots at T at a range of 5400 yards, and the shot hits at S so that angle TGS $\doteq$ 3°. Assume that angle GTS = 90°. How far from T is S?

8. Find the radius of a regular decagon, each side of which is 8 inches.

9. From a mountain top 4000 feet above a fort the angle of depression of the fort is 17°. Find the airline distance from the mountain top to the fort.



10. At a point 185 feet from the base of a tree, the angle of elevation of the top is 55°. How tall is the tree?

11. From an observation point the angles of depression of two boats in line with this point are 18° and 28°. Find the distance between the two boats if the point of observation is 4000 feet high.

12. A building stands on a horizontal plane. The angle of elevation at a certain point on the plane is 30° and at a point 100 feet nearer the building it is 45°. How high is the building?

13. Find the angles of intersection of the diagonals of a rectangle 8.3 feet wide and 13.6 feet long.

14. The area of an equilateral Δ is 300 square inches. What is the area of the inscribed circle?

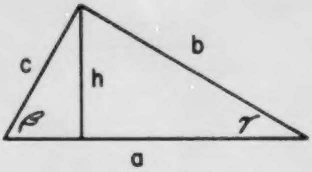
15. A circle is divided into 7 equal parts. Find the length of all possible chords whose end-points are these division points if the radius of the circle is 7 inches.

16. The minute hand of a clock is 9 inches long. At 7 minutes after 3 the line joining the ends of the hands is perpendicular to the hour hand. How long is the hour hand?

17. If the hands of a clock are 7.4 inches and 4.8 inches, at what time between 2:00 and 2:10 is the line joining the ends of the hands perpendicular to the hour hand?

18. Given Δ ABC with A, β and γ known. Let h be the altitude to a.

Prove: $h = \dfrac{a}{\cot\beta + \cot\gamma}$



19. A chord 6 inches long subtends a certain angle at the center of a circle whose radius is 5 inches. Find the length of the chord which subtends an angle twice as large.

20. The area of trapezoid ABCD is 4800 square feet. Lower base $\overline{AB}$ is 150 feet long, side $\overline{AD}$ is 47 feet long and angle A is 57°. Find the other base.

10-9. Graphs of the Trigonometric Functions.

We have found it helpful in the past to draw the graphs of the functions under study. Recall that the graph of $y = f(x)$ consists of the set of points (x,y) in the coordinate plane such that $y = f(x)$. But it is clearly impossible to draw the graphs of those trigonometric functions whose domains are the set of all signed angles, because we have no scheme for exhibiting graphically the set of all pairs $\big((A,P,\theta),\ \sin(A,P,\theta)\big)$. Notice that the first element in this pair is a signed angle (A,P,θ), which is a geometric object - not a real number.

We now recall that a second set of trigonometric functions was defined in Section 10-5. The domains of these functions are the set of all real numbers, and it is thus possible to draw their graphs in the usual way. For example, the graph of $y = \sin x$ consists of all points (x,y), where $y = \sin(O,X,x)$ and x is considered to be the radian measure of the angle (O,X,x). Similar statements hold for the graphs of the other five trigonometric functions of real numbers.

It is important to observe that the following statements are true for every x.

$$\begin{aligned}
&(i) && \sin(x + 2\pi) = \sin x \\
&(ii) && \cos(x + 2\pi) = \cos x \\
&(iii) && \sec(x + 2\pi) = \sec x \\
&(iv) && \csc(x + 2\pi) = \csc x \\
&(v) && \tan(x + \pi) = \tan x \\
&(vi) && \cot(x + \pi) = \cot x
\end{aligned}$$

Statements (i) to (iv) follow from the fact that, if x is the radian measure of an angle, $x + 2\pi$ is the radian measure of a coterminal angle. Statements (v) and (vi) follow from the facts that the angles having radian measures of x and $x + \pi$ respectively, have the same reference angle and their tangents (or cotangents) have the same algebraic sign.

If for a function $f(x)$ there exists a number p such that

$$(vii) \quad f(x + p) = f(x)$$

for all x the function f is said to be periodic. If p is the smallest positive number for which (vii) is true, the function is said to be periodic with period p. Since 2π is the smallest positive number for which statements (i) to (iv) are true for all x, we conclude on the basis of our definition that the functions sin, cos, sec and csc are all periodic with period 2π. Similarly, the tan and cot functions are periodic with period π.

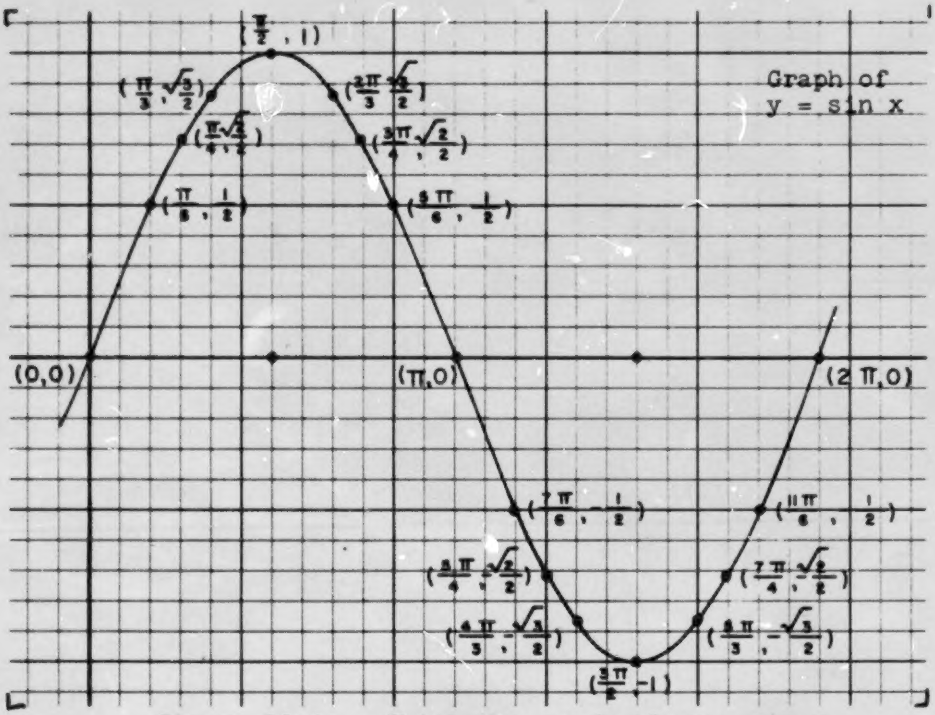


Figure 10-9a. Shows the graph of y = sin x.

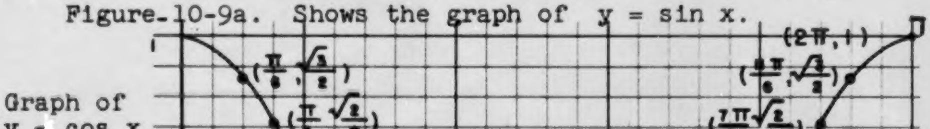


Figure 10-9b. Shows the graph of y = cos x.

Notice that the curve in Figure 10-9b is congruent to the graph of $y = \sin x$ in Figure 10-9a and is obtainable by shifting that curve $\frac{\pi}{2}$ units to the left.

Figure 10-9c shows the graph of $y = \tan x$.

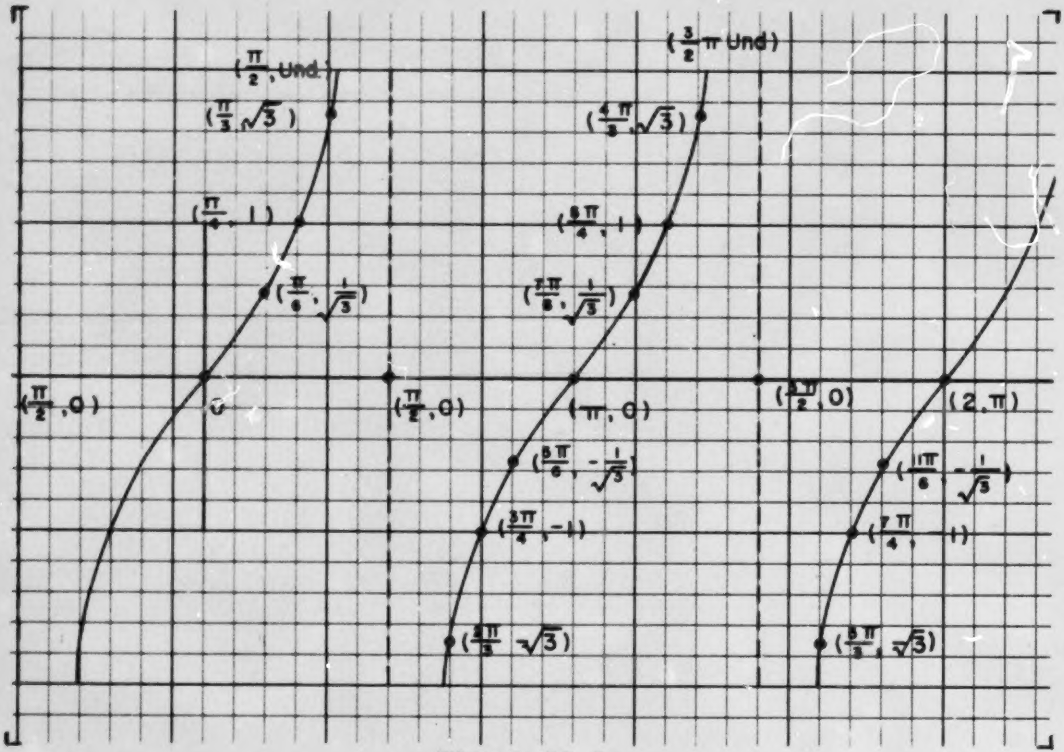


Figure 10-9c.

Notice that it is composed of congruent pieces which have the vertical lines $x = \pm \frac{\pi}{2}$, $x = \pm \frac{3\pi}{2}$ as asymptotes.

Exercises 10-9

1. Draw the graphs of each of the following sets of equations using a single set of axes.

(a) $y = \sin x$
$y = \sin 2x$

(b) $y = \cos x$
$y = 2 \cos x$

(c) $y = \tan x$
$y = \tan \frac{1}{2}x$

(d) $y = \sin \frac{1}{2}x$
$y = \cos \frac{1}{2}x$

(e) $y = \sec x$
$y = \csc x$

(f) $y = \sin x$
$y = \cos x$

(g) $y = \sin x$
$y = 2 \sin x$
$y = 3 \sin x$
$y = \frac{1}{2} \sin x$

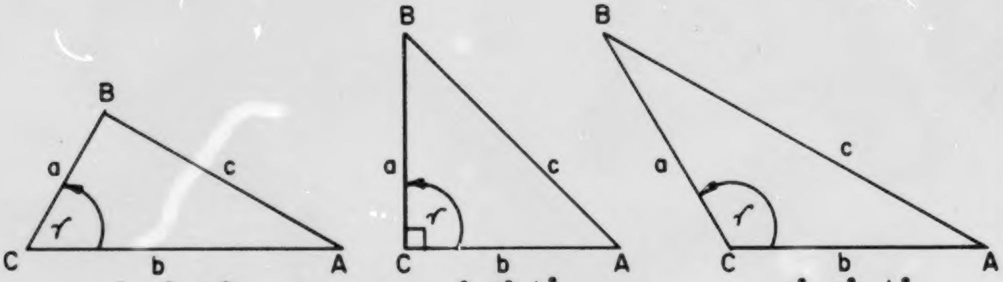
(h) $y = \sin x$
$y = \sin 2x$
$y = \sin 3x$
$y = \sin \frac{1}{2}x$

(i) $y = \sin x$
$y = \sin(x + \frac{\pi}{2})$

(j) $y = \sin(x + \frac{\pi}{2})$
$y = \cos(x - \frac{\pi}{2})$

10-10. The Law of Cosines.

One of the most famous of all mathematical theorems is the Theorem of Pythagoras, which states that in a right triangle ABC, $c^2 = a^2 + b^2$.



$c^2 < a^2 + b^2$ $\qquad$ $c^2 = a^2 + b^2$ $\qquad$ $c^2 > a^2 + b^2$

Figure 10-10a.

It is plausible that if γ is less than a right angle, then c^2 is less than $a^2 + b^2$; and if γ is greater than a right angle, then c^2 is greater than $a^2 + b^2$. Our next theorem covers all three possibilities in a single formula. It refers to any triangle ABC and uses the notation of Figure 10-10b.

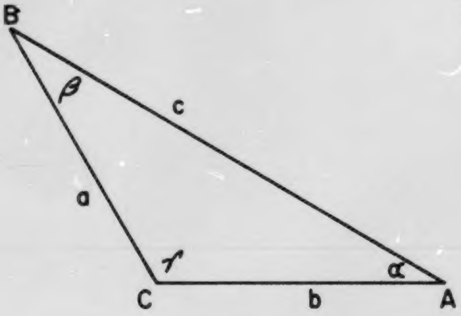


Figure 10-10b.

Theorem 10-10a. (The Law of Cosines.) In triangle ABC

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$
$$b^2 = a^2 + c^2 - 2ac \cos \beta$$
$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

Proof: We introduce a coordinate system in such a way that γ is in standard position. In this coordinate system, C has coordinates (0,0), A has coordinates (b,0), and B has coordinates which we denote by (x,y). (See Figure 10-10c). Using the distance formula we have $c^2 = (x - b)^2 + y^2 = x^2 + y^2 + b^2 - 2xb$ and $a^2 = x^2 + y^2$.

It follows that

$$c^2 = a^2 + b^2 - 2xb.$$

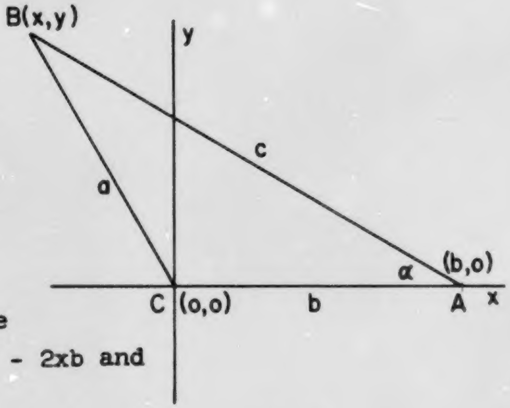


Figure 10-10c.

We also know from Theorem 10-5a that $\cos \gamma = \frac{x}{\sqrt{x^2 + y^2}}$, which is $\frac{x}{a}$. Therefore, $x = a \cos \gamma$. Substituting $a \cos \gamma$ for x gives us

$$c^2 = a^2 + b^2 - 2ab \cos \gamma .$$

The other two relations in the theorem can be proved similarly.

Example 10-10a: In triangle ABC, $a = 10$, $b = 7$, and $\gamma = 32^\circ$. Find c.

Solution: By the law of cosines

$$c^2 = 100 + 49 - 140 \cos 32^\circ .$$

Using Table 10-8a $\quad \cos 32^\circ \approx .848$ and

therefore, $\quad c^2 \approx 149 - 140(.848)$

$$\approx 30$$

Hence, $\quad c \approx 5.48$

Example 10-10b: In triangle ABC, $a = 10$, $b = 7$, and $c = 12$. Find α.

Solution: By law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha .$$

Hence, $\quad \cos \alpha = \frac{49 + 144 - 100}{2 \cdot 7 \cdot 12} = \frac{93}{168} \approx .554.$

Thus, $\quad \alpha \approx 56^\circ$ to the nearest degree.

Suppose triangle ABC is a right triangle with right angle at C, i.e., $\gamma = 90^\circ$. In this case, c is the hypotenuse of the right triangle, and since $\cos 90^\circ = 0$, the law of cosines becomes $c^2 = a^2 + b^2$. But this is just the Pythagorean Theorem. Therefore the law of cosines can be viewed as the generalization of the Pythagorean Theorem to arbitrary triangles. However, we do not have a new proof of the Pythagorean Theorem here, because our proof of the law of cosines depends on the distance formula which was established on the basis of the Pythagorean Theorem !

It is worth noting, though, that the law of cosines can be used to prove the converse of the Pythagorean Theorem. If, in triangle ABC we know that $c^2 = a^2 + b^2$, then we must show that

$\gamma = 90^\circ$. By the law of cosines $c^2 = a^2 + b^2 - 2ab \cos \gamma$ and, combining this with $c^2 = a^2 + b^2$, we obtain $\cos \gamma = 0$. We know that $0 < \gamma < 180^\circ$, and the only angle in this range whose cosine is zero is 90°. Therefore, $\gamma = 90^\circ$ as was to be proved.

Exercises 10-10

1. Use the law of cosines to solve the following:

   (a) $\alpha = 60^\circ$, $b = 10.0$, $c = 3.0$, find a.

   (b) $a = 2\sqrt{61}$, $b = 8$, $c = 10$, find γ.

   (c) $a = 4.0$, $b = 20.0$, $c = 18.0$, find α, β, and γ.

2. Find the largest angle of a triangle having sides 6, 8, and 12.

3. In the following problems find the length of the side not given.

   (a) $b = 8$, $c = 12$, $\alpha = 25^\circ$

   (b) $a = 2.5$, $b = 13$, $\gamma = 140^\circ$

   (c) $a = 60$, $c = 30$, $\beta = 40^\circ$

4. Find all three angles of the triangle in each of the following:

   (a) $a = 15$, $b = 16$, $c = 17$

   (b) $a = 24$, $b = 22$, $c = 25$

   (c) $a = 60$, $b = 30$, $c = 40$

   (d) $a = 4.5$, $b = 11$, $c = 8.5$

5. Two sides and the included angle of a parallelogram are 12 inches, 20 inches and 100° respectively. Find the length of the longer diagonal.

10-11. The Law of Sines.

The following theorem expresses the area of a triangle in terms of its sides and angles.

Theorem 10-11a. In triangle ABC

$$\begin{aligned}\text{area of triangle } ABC &= \tfrac{1}{2}ab \sin \gamma \\ &= \tfrac{1}{2}bc \sin \alpha \\ &= \tfrac{1}{2}ac \sin \beta .\end{aligned}$$

Proof: Introduce a coordinate system so that γ is in standard position. (See Figure 10-11a).

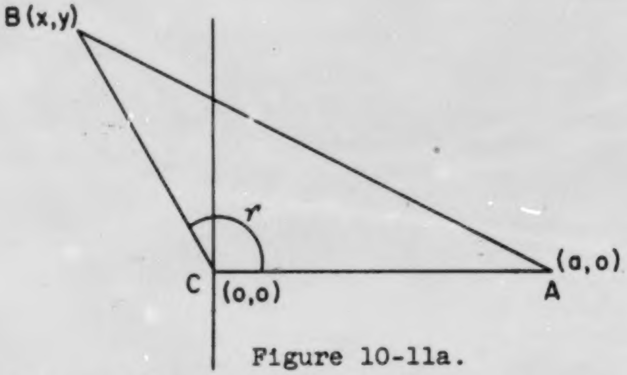


Figure 10-11a.

Then by Theorem 10-5a

$$\sin \gamma = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{a},$$

but y is also equal to h, the altitude of the triangle, so $h = a \sin \gamma$. Since the base of the triangle is b, its area is $\frac{1}{2}ab \sin \gamma$.

The other formulas follow similarly.

Theorem 10-11b. (Law of Sines). In triangle ABC,

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} .$$

Proof: According to Theorem 10-11a we have

$$\tfrac{1}{2}ab \sin\gamma = \tfrac{1}{2}bc \sin\alpha = \tfrac{1}{2}ac \sin\beta .$$

If we divide each member of these equations by $\frac{abc}{2}$ we obtain

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}.$$

Example 10-11a. If, in triangle ABC, $a = 10$, $\beta = 42^\circ$, $\gamma = 51^\circ$, find b.

Solution: Since $\alpha + \beta + \gamma = 180^\circ$ we have $\alpha = 87^\circ$. By the law of sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b},$$

or

$$b = \frac{a \sin\beta}{\sin\alpha} = \frac{10 \sin 42^\circ}{\sin 87^\circ} \approx \frac{6.69}{.999} \approx 6.7 .$$

Example 10-11b. Find the area of triangle ABC if $a = 10$, $b = 7$, $\gamma = 68^\circ$.

Solution: According to the formula in Theorem 10-11a, the area of triangle ABC $= \frac{1}{2}ab \sin\gamma = 35 \sin 68^\circ \approx 35(.927) \approx 32.4$

Example 10-11c. Are there any triangles ABC such that $b = 5$, $c = 10$, and $\gamma = 22^\circ$?

Solution: Before attempting to solve Example 10-11c let us try to construct a triangle ABC geometrically, given b, c, and γ. Lay off side $\overline{AC}$ of length b, and construct angle γ at C. Now with A as center strike an arc of radius c. There are a number of possibilities depending on b, c, and γ which are illustrated in Figure 10-11b.

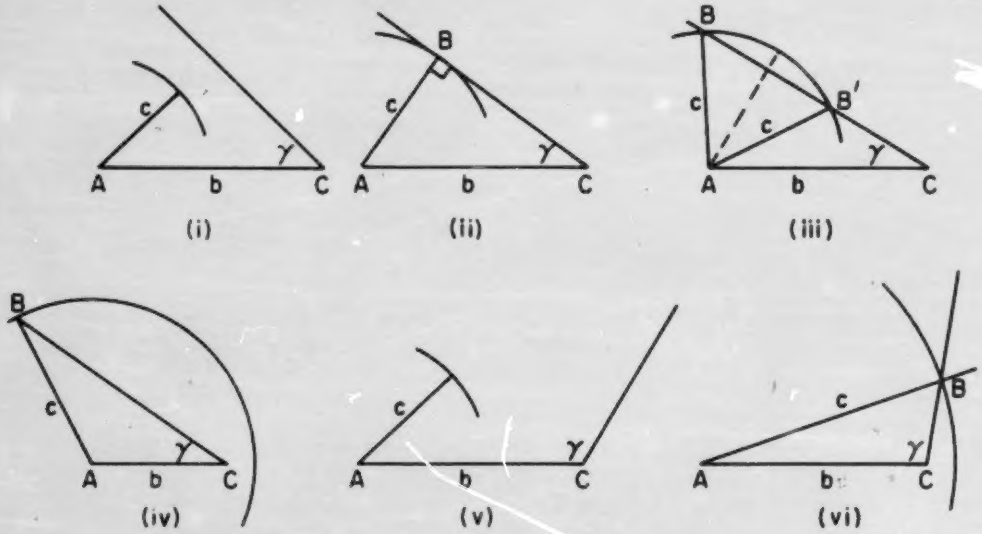

c
γ
A
b
C
(i)
B
c
γ
A
b
C
(ii)
B
B'
c
c
γ
A
b
C
(iii)
B
c
γ
A
b
C
(iv)
c
γ
A
b
C
(v)
B
c
γ
A
b
C
(vi)

Figure 10-11b.

In case (i) there is no triangle;
in case (ii) there is one triangle;
in case (iii) there are two triangles;
in case (iv) there is one triangle;
in case (v) there is no triangle;
in case (vi) there is one triangle.

Thus to solve Example 10-11c, we attempt to find β keeping in mind that there may be zero, one, or two solutions. If such a triangle exists, then by the law of sines

$$\frac{\sin\beta}{5} = \frac{\sin 22^\circ}{10}$$

or

$$\sin\beta = \frac{1}{2}\sin 22^\circ \approx .187 .$$

Recall that $\sin\beta$ is positive in the second quadrant and if $\beta = 180^\circ - \theta$ where $0 < \theta < 90^\circ$, then $\sin\beta = \sin\theta$. Thus from $\sin\beta \approx .187$ we conclude that $\beta \approx 11^\circ$ or $\beta \approx 169^\circ$ to the nearest degree. Are both of these values of β possible? If $\beta = 169^\circ$ then $\gamma + \beta = 191^\circ$ which is impossible. Why? Therefore, there is one triangle with the given data. We are in case (iv).

Example 10-11d. Are there any triangles ABC with $b = 10$, $c = 15$, and $\gamma = 105^\circ$?

Solution: We attempt to find β. If there is such a triangle, we have, from the law of sines,

$$\frac{\sin\beta}{10} = \frac{\sin\gamma}{15}.$$

But $\sin\gamma = \sin 105^\circ = \sin(180^\circ - 75^\circ) = \sin 75^\circ \approx .966$. Hence, $\sin\beta \approx \frac{2}{3}(.966) \approx .644$ and this implies $\beta \approx 40^\circ$ or $\beta \approx 140^\circ$. Clearly β can not be 140° and there is one triangle with the given data. This is an example of case (vi).

Example 10-11e. Are there any triangles ABC such that $b = 50$, $c = 10$, and $\gamma = 22^\circ$?

Solution: We attempt to find β. If there is such a triangle we have, from the law of sines,

$$\frac{\sin\beta}{50} = \frac{\sin 22^\circ}{10},$$

$$\sin\beta \approx 5(.375) > 1.$$

But we know that the sine never exceeds one, and therefore our assumption that a triangle with the given data exists leads to a contradiction. Thus there are no such triangles. This is an illustration of case (1).

Exercises 10-11

1. Use the law of sines to solve the following:

(a) $\beta = 68^\circ$, $\gamma = 30^\circ$, $c = 32.0$, find a
(b) $\alpha = 45^\circ$, $\gamma = 60^\circ$, $b = 20.0$, find c
(c) $\alpha = 26^\circ$, $\gamma = 43^\circ$, $c = 21.3$, find b
(d) $\gamma = 126^\circ$, $\alpha = 33^\circ$, $b = 3.71$, find a
(e) $\gamma = 113.2^\circ$, $\alpha = 46^\circ$, $c = 17.5$, find b
(f) $\beta = 68.5^\circ$, $\alpha = 103.2^\circ$, $c = 51.3$, find a

2. Solve completely the following triangles:

(a) $\alpha = 27^\circ$, $\gamma = 42^\circ$, $b = 24$
(b) $\gamma = 29.5^\circ$, $\beta = 48.5^\circ$, $c = 8.4$
(c) $\alpha = 132^\circ$, $\beta = 24^\circ$, $a = 135$
(d) $a = 5.8$, $\alpha = 50^\circ$, $\beta = 73^\circ$
(e) $\alpha = 102^\circ$, $\beta = 41^\circ$, $c = 52.8$
(f) $\alpha = 48.5^\circ$, $\gamma = 67.8^\circ$, $b = 28.7$

3. In each of the following, without solving, determine the number of solutions.

   (a) $\alpha = 110^\circ$, $a = 5$, $b = 4$

   (b) $\beta = 60^\circ$, $b = 12$, $c = 10$

   (c) $\gamma = 110^\circ$, $c = 36$, $b = 36$

   (d) $\alpha = 30^\circ$, $a = 8$, $b = 7$

   (e) $\alpha = 45^\circ$, $a = 14$, $b = 16$

   (f) $\alpha = 120^\circ$, $a = 12$, $b = 8$

4. In the following, determine number of solutions and solve completely.

   (a) $\alpha = 69^\circ$, $a = 5.2$, $b = 6.2$

   (b) $\beta = 13.3^\circ$, $b = 80$, $a = 193$

   (c) $\alpha = 142^\circ$, $a = 8.4$, $b = 3.7$

   (d) $\gamma = 59.6^\circ$, $a = 39$, $c = 37$

   (e) $\gamma = 5.8^\circ$, $c = 98.3$, $a = 23.2$

5. Find the area of the triangle in each of the following.

   (a) $b = 12$, $c = 14$, $\alpha = 42^\circ$

   (b) $a = 8.6$, $b = 7.9$, $\gamma = 67^\circ$

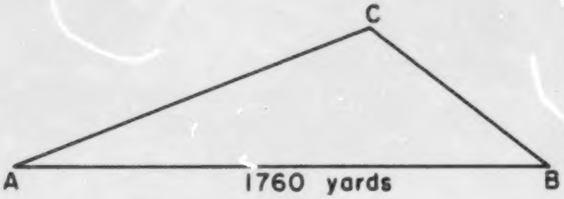
   (c) $a = 14.1$, $c = 27.4$, $\beta = 112^\circ$

   (d) $c = 5.5$, $b = 8.0$, $\alpha = 103.5^\circ$

6. One diagonal of a parallelogram is 24.3 and it makes an angle of 42.3° and 27.6° with the sides. Find the sides.

7. Two points A and B on a side of a road are 30 feet apart. A point C across the road is located so that angle CAB is 70° and angle ABC is 80°. How wide is the road?

8. Two observers, one at A and the other at B, were 1760 yards apart when they observed the flash of an enemy gun at C. If angle A was 38° and angle B was 61°, how far was each observer from the enemy gun?



9. From the top of a cliff, the angles of depression of two successive mileposts on a horizontal road running due north are 74° and 25°, respectively. Find the elevation of the cliff above the road.

10. A tower at the top of an embankment casts a shadow 125 feet long, straight down one side, when the angle of elevation of the sun is 48°. If the side of the embankment is inclined 33° from the horizontal, find the height of the tower.

11. A triangular lot has frontages 90 feet and 130 feet on two streets which intersect at an angle of 82°. Find the area of the lot.

12. The lengths of two sides of a triangular lot are 240 feet and 300 feet and the angle opposite the longer side is equal to 75°. Find the third side and the area.

10-12. The Addition Formulas.

Angle measures and trigonometric functions have a common feature, namely, they both are schemes for attaching numbers to angles. One important difference between them has to do with addition of angles. If α and β are any angles, the measure of their sum $\alpha + \beta$ is the same as the sum of the measures of α and of β. The corresponding statement is not true for trigonometric functions. For instance $\sin(30^\circ + 60^\circ) = 1$ and $\sin 30^\circ + \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2}$, which does not equal 1. In this section we derive the correct expression for $\sin(\alpha + \beta)$ and related expressions. We need a preliminary theorem.

Note: in what follows the expression $(\cos\alpha)^2$ and $(\sin\alpha)^2$ are written as $\cos^2\alpha$ and $\sin^2\alpha$ (instead of as $\cos\alpha^2$ and $\sin\alpha^2$ which could mean $\cos(\alpha)^2$ and $\sin(\alpha)^2$.)

Theorem 10-12a. Let C be a circle of radius 1, let γ be any angle whose vertex is the center of C, and let P and Q be the respective intersection of the initial and terminal sides of γ with C. Then

$$|PQ|^2 = 2 - 2\cos\gamma.$$

Proof: Introduce a coordinate system in which the initial side of γ is the positive x-axis (See Figure 10-12a).

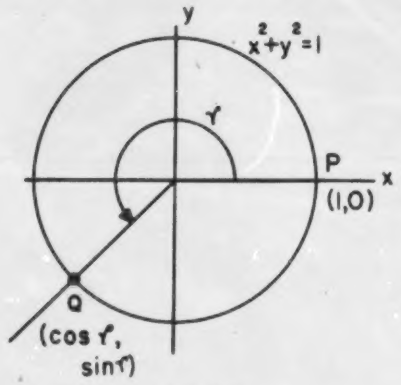


Figure 10-12a.
Length of the chord $\overline{PQ}$.

Then the coordinates of P are (1,0), and those of Q are $(\cos\gamma, \sin\gamma)$. The distance formula gives

$$|PQ|^2 = (\cos\gamma - 1)^2 + (\sin\gamma)^2$$

$$|PQ|^2 = 1 - 2\cos\gamma + \cos^2\gamma + \sin^2\gamma$$

or since

$$\cos^2\gamma + \sin^2\gamma = 1,$$

$$|PQ|^2 = 2 - 2\cos\gamma.$$

Theorem 10-12b. For all angles α and β

$$\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha.$$

Proof: We first use Theorem 10-12a to evaluate $|PQ|^2$, obtaining (see Figure 10-12b) $|PQ|^2 = 2 - 2\cos(\beta - \alpha)$. We then re-evaluate $|PQ|^2$, using the distance formula. We have

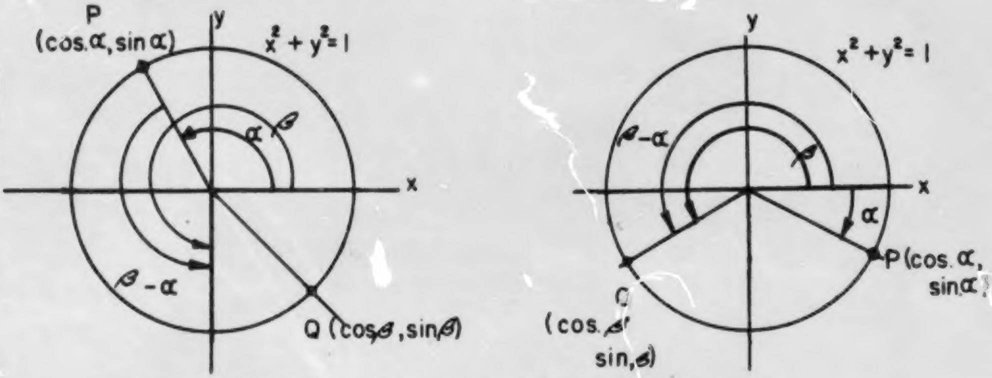


Figure 10-12b. The difference of β and α .

$$\begin{aligned}|PQ|^2 &= (\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2\\ &= \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta\\ &\quad - 2\sin\alpha\sin\beta + \sin^2\alpha\\ &= \cos^2\beta + \sin^2\beta + \cos^2\alpha + \sin^2\alpha\\ &\quad - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta.\end{aligned}$$

Since $\cos^2\beta + \sin^2\beta = \cos^2\alpha + \sin^2\alpha = 1$, we have

$$|PQ|^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

By equating this expression for $|PQ|^2$ with the one given at the beginning of this proof, we have

$$\cos(\beta - \alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha .$$

Example 10-12a. Find $\cos 15^\circ$.

Solution:
$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4} .\end{aligned}$$

We next derive similar formulas for $\sin(\beta - \alpha)$, $\cos(\alpha + \beta)$, $\sin(\alpha + \beta)$. First we need some preliminary theorems.

Theorem 10-12c. For all angles α

$$\cos(\alpha - \tfrac{\pi}{2}) = \sin\alpha ,$$

$$\sin(\alpha - \tfrac{\pi}{2}) = - \cos\alpha .$$

Proof: By Theorem 10-12b

$$\cos(\alpha - \tfrac{\pi}{2}) = \cos\alpha \cos\tfrac{\pi}{2} + \sin\alpha \sin\tfrac{\pi}{2}.$$

Since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$, it follows that

$$\cos(\alpha - \tfrac{\pi}{2}) = \sin\alpha .$$

Since this relation holds for any angle α, we can use it for $\alpha - \frac{\pi}{2}$ itself. It then reads

$$\cos\left((\alpha - \tfrac{\pi}{2}) - \tfrac{\pi}{2}\right) = \sin(\alpha - \tfrac{\pi}{2}).$$

The left hand side of this equation is $\cos(\alpha - \pi)$, which equals $\cos\alpha \cos\pi + \sin\alpha \sin\pi$. Since $\cos\pi = -1$ and $\sin\pi = 0$, we conclude that

$$\sin(\alpha - \tfrac{\pi}{2}) = -\cos\alpha .$$

Theorem 10-12d. For all angles α and β

$$\sin(\beta - \alpha) = \sin\beta \cos\alpha - \cos\beta \sin\alpha .$$

Proof: By Theorem 10-12c, we have

$$\begin{aligned} \sin(\beta - \alpha) &= \cos\left((\beta - \alpha) - \tfrac{\pi}{2}\right) \\ &= \cos\left((\beta - \tfrac{\pi}{2}) - \alpha\right). \end{aligned}$$

Using Theorem 10-12b, we can write

$$\cos\left((\beta - \tfrac{\pi}{2}) - \alpha\right) = \cos(\beta - \tfrac{\pi}{2}) \cos\alpha + \sin(\beta - \tfrac{\pi}{2}) \sin\alpha .$$

We substitute $\sin\beta$ for $\cos(\beta - \frac{\pi}{2})$ and $-\cos\beta$ for $\sin(\beta - \frac{\pi}{2})$ in this last relation to obtain

$$\sin(\beta - \alpha) = \sin\beta \cos\alpha - \cos\beta \sin\alpha .$$

Example 10-12b. Find $\sin 15^\circ$.

Solution:

$$\begin{aligned} \sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}. \end{aligned}$$

Theorem 10-12e. For all angles α

$$\cos(-\alpha) = \cos\alpha$$

$$\sin(-\alpha) = -\sin\alpha .$$

Proof:

$$\begin{aligned} \cos(-\alpha) &= \cos(0 - \alpha) \\ &= \cos 0^\circ \cos\alpha + \sin 0^\circ \sin\alpha . \end{aligned}$$

Since $\cos 0^\circ = 1$, $\sin 0 = 0$, we conclude that

$$\cos(-\alpha) = \cos\alpha .$$

Since

$$\begin{aligned} \sin(-\alpha) &= \sin(0 - \alpha) \\ &= (\sin 0)(\cos\alpha) - (\cos 0)(\sin\alpha) \end{aligned}$$

we have

$$\begin{aligned} \sin(-\alpha) &= 0 \cdot \cos\alpha - 1 \cdot \sin\alpha \\ &= -\sin\alpha . \end{aligned}$$

Theorem 10-12f. For all angles α and β

$$\cos(\beta + \alpha) = \cos\beta \cos\alpha - \sin\beta \sin\alpha$$

$$\sin(\beta + \alpha) = \sin\beta \cos\alpha + \cos\beta \sin\alpha .$$

Proof:
$$\begin{aligned}\cos(\beta + \alpha) &= \cos\big(\beta - (-\alpha)\big)\\ &= \cos\beta \cos(-\alpha) + \sin\beta \sin(-\alpha)\\ &= \cos\beta \cos\alpha - \sin\beta \sin\alpha .\end{aligned}$$

$$\begin{aligned}\sin(\beta + \alpha) &= \sin\big(\beta - (-\alpha)\big)\\ &= \sin\beta \cos(-\alpha) - \cos\beta \sin(-\alpha)\\ &= \sin\beta \cos\alpha + \cos\beta \sin\alpha .\end{aligned}$$

Example 10-12c. Find $\cos 75^\circ$ and $\sin 75^\circ$.

Solution:
$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \sin 30^\circ \cos 45^\circ\\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6} + \sqrt{2}}{4}.\end{aligned}$$

Notice that $\sin 75^\circ = \cos 15^\circ$ and $\cos 75^\circ = \sin 15^\circ$. This illustrates Theorem 10-12c.

Theorem 10-12g. For all angles α

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha .$$

Proof:

$$\begin{aligned}\sin 2\alpha &= \sin(\alpha+\alpha)\\ &= \sin\alpha\cos\alpha + \sin\alpha\cos\alpha\\ &= 2\sin\alpha\cos\alpha .\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos(\alpha+\alpha)\\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha\\ &= \cos^2\alpha - \sin^2\alpha .\end{aligned}$$

Summary of Formulas

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\
\cos(\alpha-\tfrac{\pi}{2}) &= \sin\alpha\\
\sin(\alpha-\tfrac{\pi}{2}) &= -\cos\alpha\\
\cos(-\alpha) &= \cos\alpha\\
\sin(-\alpha) &= -\sin\alpha\\
\sin 2\alpha &= 2\sin\alpha\cos\alpha\\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha
\end{aligned}$$

Exercises 10-12

1. Let α be an angle in the third quadrant whose cosine is $-\frac{4}{5}$ and β be an angle in second quadrant whose tangent is $-\frac{5}{12}$. Find

   (a) $\sin(\alpha+\beta)$

   (b) $\cos(\alpha+\beta)$

   (c) $\sin(\alpha-\beta)$

   (d) $\cos(\alpha-\beta)$

   (e) $\tan(\alpha+\beta)$

   (f) $\tan(\alpha-\beta)$

2. Use the addition formulas to compute the exact value of the following:

   (a) $\sin 75^\circ$
   (b) $\cos 75^\circ$
   (c) $\tan 75^\circ$
   (d) $\sin 15^\circ$
   (e) $\cos 105^\circ$
   (f) $\sin 195^\circ$

3. Use the addition formulas to find the exact value of the following:

   (a) $\cos(\pi - \frac{\pi}{3})$
   (b) $\sin(\pi - \frac{\pi}{6})$
   (c) $\cos(\pi + \frac{\pi}{3})$
   (d) $\sin(\frac{3\pi}{2} + \frac{\pi}{4})$

4. Show that $\cos(\alpha - \frac{\pi}{2}) = \sin \alpha$ for

   (a) $\alpha = 45^\circ$
   (b) $\alpha = 210^\circ$
   (c) $\alpha = 180^\circ$
   (d) $\alpha = \frac{\pi}{3}$
   (e) $\alpha = \frac{3\pi}{4}$
   (f) $\alpha = -\frac{3\pi}{2}$

5. Show that $\sin(\alpha - \frac{\pi}{2}) = -\cos \alpha$ for

   (a) $\alpha = 60^\circ$
   (b) $\alpha = 150^\circ$
   (c) $\alpha = 300^\circ$
   (d) $\alpha = \frac{2\pi}{3}$
   (e) $\alpha = \frac{5\pi}{4}$
   (f) $\alpha = \frac{11\pi}{6}$

6. Prove $\cos 2\alpha = 2\cos^2 \alpha - 1$, and deduce from this equation the half angle formula $\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$.

7. Prove $\cos 2\alpha = 1 - 2\sin^2 \alpha$, and deduce from this equation the half angle formula $\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$.

8. Compute the exact value of $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$ for the following:

   (a) $\cos \alpha = \frac{3}{5}$, α in quadrant I

   (b) $\tan \alpha = \frac{4}{3}$, α in quadrant III

   (c) $\sin \alpha = \frac{2}{3}$, α in quadrant II

   (d) $\cos \alpha = \frac{3}{4}$, α in quadrant IV

9. Compute the exact value of $\sin \frac{\alpha}{2}$, $\cos \frac{\alpha}{2}$, and $\tan \frac{\alpha}{2}$ for the following:

   (a) $\cos \alpha = \frac{1}{2}$, α in quadrant IV

   (b) $\sin \alpha = -\frac{3}{5}$, α in quadrant III

   (c) $\cos \alpha = -\frac{5}{13}$, α in quadrant II

   (d) $\sin \alpha = \frac{12}{13}$, α in quadrant I

10. Use the formulas from Problems 6 and 7 to compute the exact value of

    (a) $\cos 15^\circ$ (c) $\sin 11.25^\circ$

    (b) $\cos 22.5^\circ$ (d) $\sin 7.5^\circ$

10-13. Identities and Equations.

Equations such as

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$

are known as identities. They yield true statements no matter what angle or real number is substituted for α. In a slightly generalized sense, the following equation is an identity.

10-13a $\tan \theta = \frac{\sin \theta}{\cos \theta}$ $\theta \neq (2k + 1)\frac{\pi}{2}$

since, by Theorem 10-5a,

$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan \theta.$$

This identity has one peculiarity which should be observed carefully; $\tan \theta$ is not defined for $\theta = (2k + 1)\frac{\pi}{2}$ and $\frac{\sin \theta}{\cos \theta}$ is not defined for $\theta = (2k + 1)\frac{\pi}{2}$ since $\cos(2k + 1)\frac{\pi}{2} = 0$. Thus, the two sides of equation 10-13a are equal for every value of θ for which the two sides are defined, and $\tan \theta = \frac{\sin \theta}{\cos \theta}$ is also called an identity.

The equation

$$\sin 2\alpha = 2 \sin \alpha$$

yields a true statement if α is replaced by $2n\pi$, n an integer, but it yields a false statement for every other value of α. An equation of this type is called a conditional equation. We have mathematical responsibilities toward each of these types of equations. We shall be asked to prove identities, that is, prove that the solution set consists of all values of the variable. More precisely, to prove an identity means to prove that the solution set consists of all values of the variable for which the two sides of the equation are defined. To solve a conditional equation means to find the solution set.

There are no standardized methods for proving identities or solving equations. To prove an identity or to solve an equation often requires ingenuity and perseverance, and many methods must be devised to handle all the problems that arise. The procedures are best explained by a variety of examples.

Example 10-13a. Prove the identity

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} .$$

Solution: Observe that neither side of this equation is defined for

$$\alpha = \frac{\pi}{4} + k \cdot \frac{\pi}{2} , \quad k \text{ an integer,}$$

for, on the left, 2α is an angle co-terminal with $\frac{\pi}{2}$ and the tangent is undefined; on the right, $\tan^2 \alpha = 1$ and the denominator vanishes. We are thus asked to prove

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \quad \alpha \neq \frac{\pi}{4} + k \cdot \frac{\pi}{2}.$$

By the proof at the beginning of this Section,

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} , \quad \alpha \neq \frac{\pi}{4} + k \cdot \frac{\pi}{2}$$

$$= \frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha - \sin^2 \alpha} \quad \text{(by the formulas from Section 10-12)}$$

$$= \frac{2 \frac{\sin \alpha}{\cos \alpha}}{1 - \frac{\sin^2 \alpha}{\cos^2 \alpha}} \quad \text{(divide the numerator and denominator by } \cos^2 \alpha \text{)}$$

$$= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} , \alpha \neq \frac{\pi}{4} + k \cdot \frac{\pi}{2}.$$

Example 10-3b. Prove the identity

$$\tan(\theta + \pi) = \tan \theta \quad , \quad \theta \neq (2k + 1)\frac{\pi}{2}$$

Solution: By Equation 10-13a,

$$\begin{aligned}\tan(\theta + \pi) &= \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)}, \qquad \theta \neq (2k + 1)\frac{\pi}{2}\\ &= \frac{-\sin \theta}{-\cos \theta} \qquad \text{(by the formulas in Section 10-12)}\\ &= \frac{\sin \theta}{\cos \theta}\\ &= \tan \theta.\end{aligned}$$

Example 10-13c. Prove the following identity

$$\frac{\sin \alpha}{1 + \cos \alpha} = \tan \frac{\alpha}{2}, \qquad \alpha \neq (2k + 1)\pi.$$

Solution: The key to the solution is the observation that $\alpha = 2(\frac{\alpha}{2})$. Thus

$$\begin{aligned}\frac{\sin \alpha}{1 + \cos \alpha} &= \frac{\sin 2(\frac{\alpha}{2})}{1 + \cos 2(\frac{\alpha}{2})}, \qquad \alpha \neq (2k + 1)\pi\\ &= \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{1 + \cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}} \qquad \text{(by the identities in Section 10-12)}\\ &= \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}}\\ &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}\\ &= \tan \frac{\alpha}{2}\end{aligned}$$

Example 10-13d. Prove the following identity:

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.$$

Solution: The simplest proof of this identity employs a device. Observe that

$$\alpha = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}$$

$$\beta = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2} .$$

Then by the addition formulas in Section 10-12

$$\sin\alpha = \sin\left[\left(\frac{\alpha+\beta}{2}\right) + \left(\frac{\alpha-\beta}{2}\right)\right]$$

$$\sin\beta = \sin\left[\left(\frac{\alpha+\beta}{2} - \left(\frac{\alpha-\beta}{2}\right)\right]$$

$$\begin{aligned}\sin\alpha + \sin\beta &= \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\ &\quad + \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \\ &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) .\end{aligned}$$

Example 10-13e. Find all solutions of the following equation:

$$\sin x = 2\cos x.$$

Solution: Observe first that $x = (2k+1)\frac{\pi}{2}$ is not a solution of the given equation. Then $\cos x \neq 0$ for a value of x which is a solution of the equation, and the given equation is equivalent to the equation

$$\frac{\sin x}{\cos x} = 2 \quad , \qquad \cos x \neq 0 \quad ,$$

or

$$\tan x = 2.$$

Interpolation in Table 10-8a, shows that x is 1.107 radians approximately. From Example 10-13b above, it follows that $\pi + 1.107$ radians, or 4.249 radians, is also a solution. Finally, since the trigonometric functions are periodic with period 2π, all solutions of the given equation are

$$x \approx 1.107 \pm 2k\pi \text{ radians}$$

$$x \approx 4.249 \pm 2k\pi \text{ radians}$$

where k is an integer.

Example 10-13f. Find all solutions of the following equation:

$$2 \sin^2 \theta - 3 \sin \theta + 1 = 0.$$

Solution: It should be observed first that the given equation is a quadratic equation in $\sin \theta$. It would be possible to solve for $\sin \theta$ by using the formula for the roots of a quadratic equation, but it is simpler in the present case to solve by factoring. The given equation is equivalent to

$$(2 \sin \theta - 1)(\sin \theta - 1) = 0,$$

and all solutions can be found by solving the two simpler equations

$$2 \sin \theta - 1 = 0, \quad \sin \theta - 1 = 0.$$

The solutions of the given equation are thus,

$$\theta = \frac{\pi}{6} + 2k\pi$$

$$\theta = \frac{5\pi}{6} + 2k\pi$$

$$\theta = \frac{\pi}{2} + 2k\pi$$

where k is an integer.

Example 10-13g. Solve the equation

$$\tan x = 2x.$$

Solution: By scanning the entries in Table 10-8a, we see that for small values of x,

$$\tan x < 2x,$$

whereas for large values of x,

$$\tan x > 2x.$$

The change in the direction of the inequality occurs between $x = 1.152$ and $x = 1.169$, that is,

$$2.246 = \tan 1.152 < 2(1.152) = 2.304,$$

$$2.356 = \tan 1.169 > 2(1.169) = 2.338.$$

Since 2x and tan x are continuous, it follows that there is a solution of the equation between $x = 1.152$ and $x = 1.169$ radians. Methods are given in more advanced courses for approximating this solution to as many decimal places as may be desired.

There are graphical methods which are useful in finding the approximate values of the solutions of trigonometric equations. The graphical solution in the present case shows that the given equation has an infinite number of solutions. Figure 10-13a shows the graphs of $y = 2x$ and $y = \tan x$. If (x_0, y_0) is a point of intersection of the graphs of these two equations, then

$$y_0 = 2x_0$$
$$y_0 = \tan x_0,$$

and $2x_0 = \tan x_0$. Thus x_0 is a solution of the given equation.

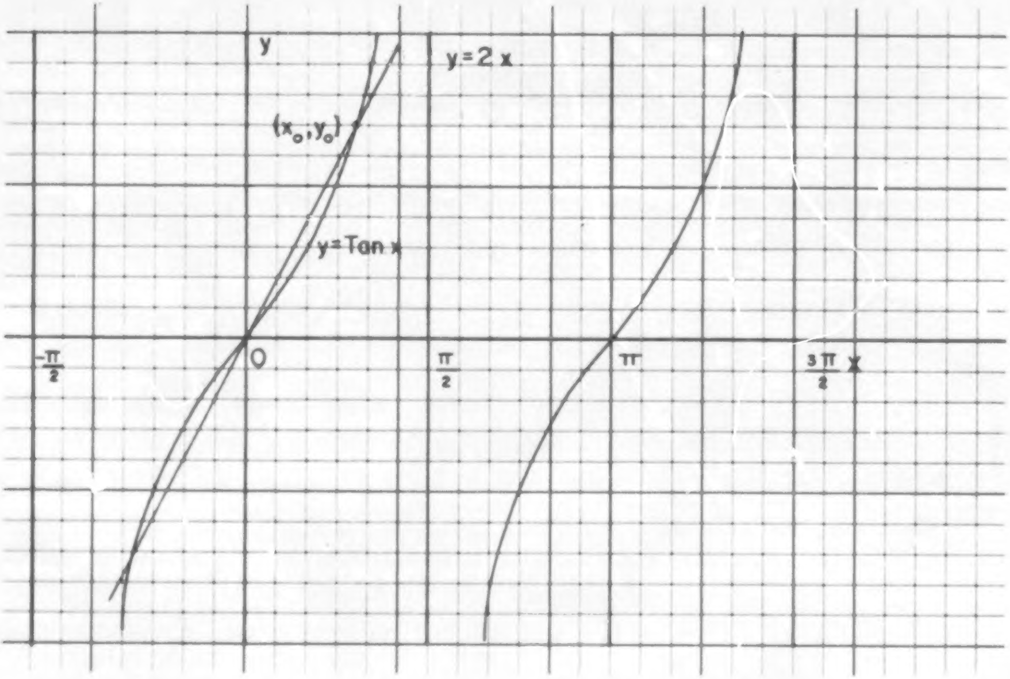


Figure 10-13a. Graphical solution of $\tan x = 2x$.

It is clear from the figure that the line $y = 2x$ intersects the graph of $y = \tan x$ in infinitely many points. For large values of x the intersections are almost on the lines $x = (2k + 1)\frac{\pi}{2}$, and $x = (2k + 1)\frac{\pi}{2}$ is approximately a solution if k is an integer whose absolute value is large.

Exercises 10-13a

Prove the following identities:

1. $\tan \theta \cos \theta = \sin \theta$
2. $(1 - \cos \theta)(1 + \cos \theta) = \sin^2 \theta$
3. $\dfrac{\cos \theta}{1 + \sin \theta} = \dfrac{1 - \sin \theta}{\cos \theta}$
4. $\tan \theta = \dfrac{\sin 2 \theta}{1 + \cos 2 \theta}$
5. $\dfrac{2}{\csc^2 x} = 1 - \dfrac{1}{\sec 2x}$
6. $2 \csc 2 \theta = \sec \theta \csc \theta$
7. $\tan \theta \sin 2 \theta = 2 \sin^2 \theta$
8. $1 - 2 \sin^2 \theta + \sin^4 \theta = \cos^4 \theta$
9. $\dfrac{2 \cos^2 \theta - \sin^2 \theta + 1}{\cos \theta} = 3 \cos \theta$
10. $\sin \theta \tan \theta + \cos \theta = \dfrac{1}{\cos \theta}$
11. $\dfrac{1}{\cos^2 \theta} + \tan^2 \theta + 1 = \dfrac{2}{\cos^2 \theta}$
12. $\sin^4 \theta - \sin^2 \theta \cos^2 \theta - 2 \cos^4 \theta = \sin^2 \theta - 2 \cos^2 \theta$
13. $\dfrac{\cos^4 \theta - \sin^4 \theta}{1 - \tan^4 \theta} = \cos^4 \theta$
14. $\sec^2 \theta - \csc^2 \theta = (\tan \theta + \cot \theta)(\tan \theta - \cot \theta)$
15. $\tan x - \tan y = \sec x \sec y \sin(x - y)$

16. $\sin 4\theta = 4 \sin\theta \cos\theta \cos 2\theta$

17. $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

18. $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$

19. $\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

20. $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$

21. $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

22. $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$

23. $\dfrac{\sin\theta}{1+\cos\theta} = \tan\dfrac{\theta}{2}$

24. $3\sin\theta - \sin 3\theta = 4\sin^3\theta$

25. Prove that none of the following is an identity by counter example. See Section 10-7, Problem 6.

(a) $\cos(\alpha-\beta) = \cos\alpha - \cos\beta$

(b) $\cos(\alpha+\beta) = \cos\alpha + \cos\beta$

(c) $\sin(\alpha-\beta) = \sin\alpha - \sin\beta$

(d) $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$

(e) $\cos 2\alpha = 2\cos\alpha$

(f) $\sin 2\alpha = 2\sin\alpha$

26. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

27. $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{1-\cos 2\theta}{\sin 2\theta}$

28. $\dfrac{\csc\theta - 1}{\cot\theta} = \dfrac{\cot\theta}{\csc\theta + 1}$

29. If $A + B + C = 180^\circ$, prove

(a) $\sin A = \sin(B + C)$

(b) $\cos A = -\cos(B + C)$

Exercises 10-13b

Solve the following equations for $0 \leqq \theta \leqq 2\pi$

1. $2 \sin \theta - 1 = 0$
2. $4 \cos^2 \theta - 3 = 0$
3. $3 \tan^2 \theta - 1 = 0$
4. $\sin^2 \theta - \cos^2 \theta + 1 = 0$
5. $2 \cos^2 \theta - \sqrt{3} \cos \theta = 0$
6. $\sec^2 \theta - 4 \sec \theta + 4 = 0$
7. $3 \sec \theta + 2 = \cos \theta$
8. $4 \sin^3 \theta - \sin \theta = 0$
9. $2 \sin^2 \theta - 5 \sin \theta + 2 = 0$
10. $2 \sin \theta \cos \theta + \sin \theta = 0$
11. $\sqrt{3} \csc^2 \theta + 2 \csc \theta = 0$
12. $2 \sin^2 \theta + 3 \cos \theta - 3 = 0$
13. $\cos 2 \theta = 0$
14. $4 \tan^2 \theta - 3 \sec^2 \theta = 0$
15. $\cos 2 \theta - \sin \theta = 0$
16. $2 \cos^2 \theta + 2 \cos 2 \theta = 1$
17. $\cos 2 \theta + 2 \cos^2 \frac{\theta}{2} = 1$
18. $\sec^2 \theta - 2 \tan \theta = 0$
19. $\sin 2 \theta - \cos^2 \theta + 3 \sin^2 \theta = 0$
20. $\cos 2 \theta - \cos \theta = 0$
21. $\cos 2 \theta \cos \theta + \sin 2 \theta \sin \theta = 1$
22. $\cos^2 \theta - \sin^2 \theta = \sin \theta$
23. $2 \sin^2 \theta - 3 \cos \theta - 3 = 0$
24. $\cos \theta = \dfrac{1 + \cos^2 \theta}{2}$

25. $\cot \theta + 2 \sin \theta = \csc \theta$

26. $\cos \theta + \sin \theta = 0$

27. $3 \sin \theta + 4 \cos \theta = 0$

28. Prove that if k is any real number then the equation $\sin x = k \cos x$ has a solution.

29. $\tan \theta = \theta$

30. $\pi \sin \theta = 2\theta$

10-14. Miscellaneous Exercises.

1. Convert each of the following to radians:

   (a) 0°

   (b) 90°

   (c) 60°

   (d) 100°

   (e) 390°

   (f) 1000°

   (g) 1°

   (h) -100°

   (i) -1000°

   (j) $\frac{12}{5\pi}^\circ$

   (k) $\frac{9}{2}^\circ$

   (ℓ) $\frac{180}{\pi}^\circ$

   (m) $\frac{\pi}{180}^\circ$

2. Convert each of the following to degrees:

   (a) 0 radians

   (b) π radians

   (c) $\frac{\pi}{2}$ radians

   (d) $\frac{\pi}{6}$ radians

   (e) 10π radians

   (f) 1 radian

   (g) -1 radian

   (h) 2 radians

   (i) -10 radians

   (j) $\frac{5\pi}{12}$ radians

   (k) $\frac{180}{\pi}$ radians

   (ℓ) $\frac{\pi}{180}$ radians

   (m) 90 radians

3. Angles are sometimes measured in revolutions, where 1 revolution is 2π radians, and also in mils where 3200 mils is π radians. For each of these units, find the radius of a circle for which a unit angle corresponds to a unit distance on the circumference?

4. Using the definitions in Problem 3, convert:
   (a) 10,000 mils to revolutions
   (b) 108 degrees to mils
   (c) 10,000 mils to degrees
   (d) 108 degrees to revolutions
   (e) 10,000 degrees to mils
   (f) .8 revolutions to degrees
   (g) 80 degrees to revolutions
   (h) .8 radians to degrees
   (i) 80 mils to radians
   (j) 800 mils to revolutions

5. Find $\sin \theta$, $\cos \theta$ and $\tan \theta$ if the terminal side of θ, in its standard position, goes through the given point.
   (a) $(-3,4)$ (d) $(-3,-2)$
   (b) $(-2,0)$ (e) $(3,-5)$
   (c) $(2,5)$

6. Sketch in standard position all the angles between 0° and 360° which satisfy the following conditions, and give values of the other functions of these angles.
   (a) $\sin \theta = \frac{4}{5}$ (c) $\tan \theta = -\frac{1}{2}$
   (b) $\cos \theta = -\frac{3}{7}$

7. Express the following as functions of positive acute angles.

(a) $\cos 170^\circ$

(b) $\sin 160^\circ$

(c) $\cos(-130^\circ)$

(d) $\sin 640^\circ$

(e) $\tan(-45^\circ)$

(f) $\cos 305^\circ$

(g) $\cos(-100^\circ)$

(h) $\sin \frac{4}{9}\pi$

(i) $\cos -\frac{2}{15}\pi$

(j) $\tan \frac{7}{3}\pi$

8. If $\sin\alpha = \frac{1}{3}$ and $\sin\beta = \frac{1}{4}$, (α and β each are acute angles) find

(a) $\sin(\alpha + \beta)$

(b) $\sin(\alpha - \beta)$

(c) $\cos(\alpha + \beta)$

(d) $\cos(\alpha - \beta)$

(e) $\sin 2\alpha$

(f) $\cos 2\beta$

9. Find the value of the following:

(a) $\sin 90^\circ + \cos 120^\circ + \tan 225^\circ + \cos 180^\circ$

(b) $\sin 30 \cos 150 - \sin 60 \cos 45$

(c) $\sin 330^\circ \tan 135^\circ - \sin 225^\circ \cos 300^\circ \tan 180^\circ$

10. Solve the following triangles for the indicated parts. Given:

(a) $a = 3$, $b = 2$, $\gamma = 60^\circ$, find c.

(b) $a = 5$, $b = 6$, $c = 7$, find β.

(c) $c = 16$, $\beta = 84^\circ$, $\gamma = 54^\circ$, find a.

(d) $c = 5\sqrt{6}$ $\alpha = 45^\circ$, $a = 6$, find b.

(e) $a = 20$, $b = 21$, $\gamma = 43^\circ 35'$, find c.

(f) $b = 5$, $\alpha = 75^\circ$, $\beta = 30^\circ$, find c and a.

(g) $\alpha = 60^\circ$ $a = 8\sqrt{3}$, $c = 15$, find β.

(h) $b = 15$, $c = 2$, $\alpha = 36^\circ$, find Area.

(i) $a = 12$, $b = 35$, $c = 37$, find γ.

(j) $a = 21$, $b = 17$, $c = 10$, find Area.

11. Prove that $\tan(-\theta) = -\tan\theta$

PROVE

12. $\cos(\frac{\pi}{2} - \theta) = \sin\theta$

13. $\sin(2\pi - \theta) = -\sin\theta$

14. $\cos\theta\cos 2\theta - \sin\theta\sin 2\theta = \cos 3\theta$

15. $\cos 2\theta\cos\theta + \sin 2\theta\sin\theta = \cos\theta$

16. $2\cos^2\frac{\theta}{2} - \cos\theta = 1$

17. $2\sin\theta + \sin 2\theta = \frac{2\sin^3\theta}{1 - \cos\theta}$

18. $(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$

19. $4\sin^2\theta\cos^2\theta = 1 - \cos^2 2\theta$

20. $-\cos^2\theta = \frac{\cos^2 2\theta - 1}{4\sin^2\theta}$

21. $\cos x + \sin x = \frac{\cos 2x}{\cos x - \sin x}$

Find all primary angles which are solutions of the following equations.

22. $\sin x - \tan x = 0$

23. $1 - \sin^2 x = \cos x$

24. $\cos x = \frac{1 - \cos x}{2}$

25. $\sin 2\theta - \sin\theta = 0$

26. $\cos 2\theta = 2 - 2\cos^2\frac{\theta}{2}$

27. $\cos 3\theta - \cos\theta = 0$

28. $2\cos^2 2\theta - 2\sin^2 2\theta = 1$

29. $2\cos^2\theta - \sin\theta - 1 = 0$

30. $\frac{1 - \cos\theta}{\sin\theta} = \sin\theta$

31. $\cot^2\theta + \csc\theta = 1$

32. Let a and b be any non-zero real numbers and let θ be any angle, prove that there is an angle α such that

$$a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos(\theta - \alpha).$$

33. In a triangle, one angle is 36° and another is $\frac{2\pi}{3}$ radians. Find the third angle in radians.

34. Through how many radians does the minute hand of a clock revolve in 40 minutes.

35. Find the three angles of a triangle ABC, given $a = 200$, $b = 300$, and $c = 400$.

36. Find the remaining parts of the triangle ABC, given $b = 128$, $c = 145$ and $\beta = 21^\circ$.

37. A man standing 152 feet from the foot of a flagpole, which is on his eye level, observes that the angle of elevation of the top of the flagpole is 48°. Find the height of the pole.

38. Two points A and B are on the bank of a river are 40 feet apart. A point C across the river is located so that angle CAB is 70° and angle ACB is 70°. How wide is the river?

39. The adjacent sides of a parallelogram are 20 and 15 inches, respectively, while the shorter diagonal is 17 inches. What is the length of the longer diagonal.

40. A flagstaff known to be 20 feet high stands on top of a building. An observer across the street observes that the angle of elevation of the bottom of the flagstaff is 69° and that the angle of elevation of the top of the flag is 76°. Find the height of the building.

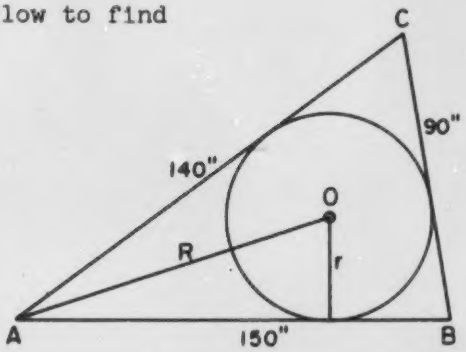
41. $\overline{AB}$ is a tower which stands on a vertical cliff $\overline{BC}$. At a point P 310 feet from the foot of the cliff, the angle of elevation of B is 21° and the angle of elevation of A is 35°. Find the height of the tower.

42. Use the figure below to find the following:

a. R,

b. r,

c. $\angle$ BAO.


C
90"
140"
O
R
r
A
150"
B

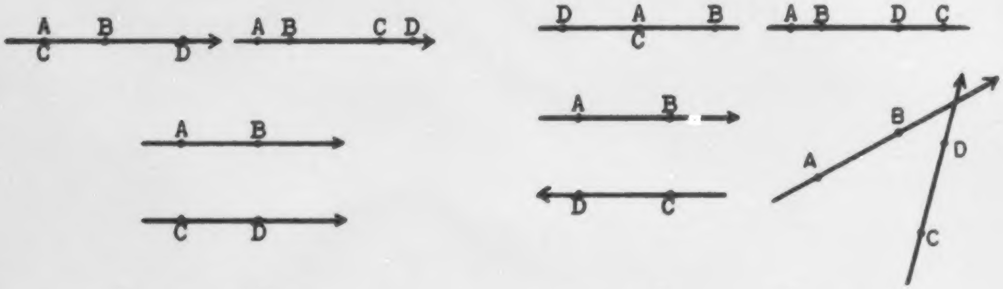
Chapter 11

THE SYSTEM OF VECTORS

11-1. Directed Line Segments.

It is assumed in this chapter that you are familiar with plane geometry. We review some of the symbols of geometry. $\overleftrightarrow{AB}$ means the line which contains the distinct points A and B . $\overrightarrow{AB}$ means the ray whose vertex is A and which also contains the point B . $|AB|$ means the distance from A to B (and from B to A). It is a positive real number if A and B are distinct. It is zero if A and B are the same.

We need one further idea which is not ordinarily covered in geometry--that of parallel rays. Rays are said to be parallel if they lie on lines which are either parallel or coincident, and if they are similarly sensed. Figure 11-1a shows typical instances of rays which are parallel and of rays which are not parallel, and is supposed to take the place of a formal definition.



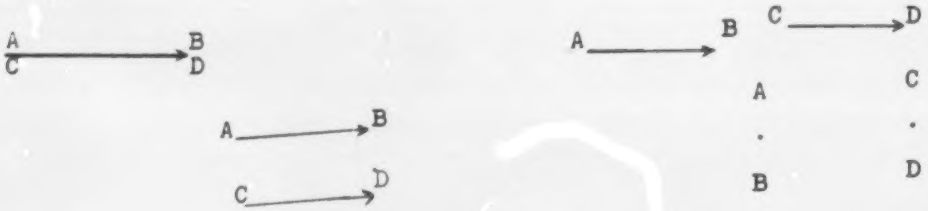
rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are $||$

rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not $||$

Fig. 11-1a

Definition 11-1a: A line segment is said to be a directed line segment if one of its endpoints is designated as its initial point and the other endpoint is designated as its terminal point. We use the symbol $\overrightarrow{AB}$ to denote the directed line segment whose initial point is A and whose terminal point is B . We say that directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if it is true that their lengths are the same and also that the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel. We write $\overrightarrow{AB} \doteq \overrightarrow{CD}$ to denote the fact that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent.

Note: We consider that a single point can be both initial and terminal point of the same directed line segment and we consider that all such directed line segments are equivalent to one another.

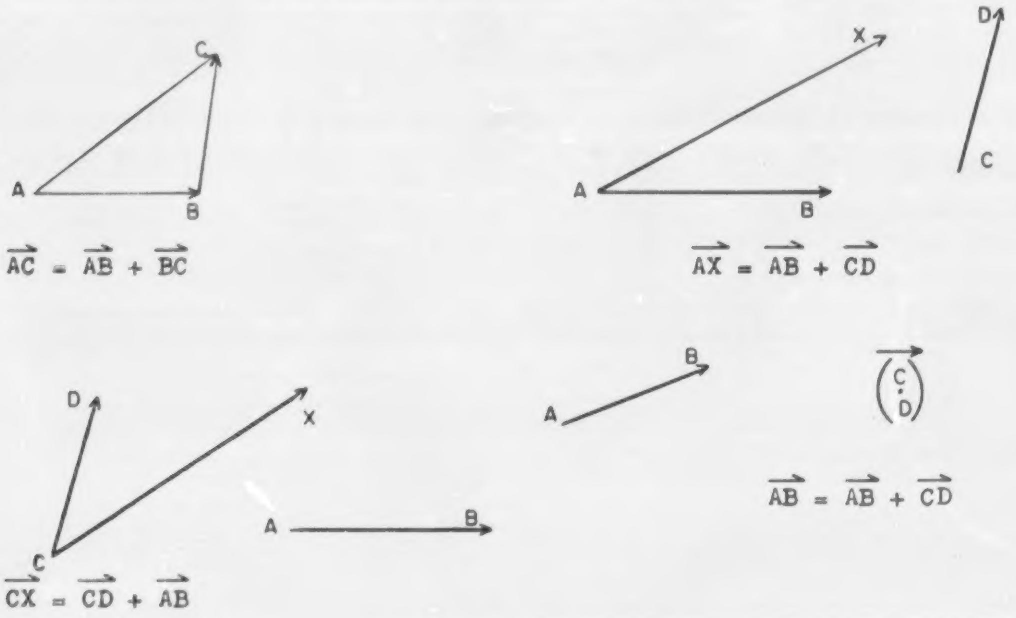


$\overrightarrow{AB} \doteq \overrightarrow{CD}$

Fig. 11-1b

Figure 11-1b shows some pairs of equivalent directed line segments. It uses the convention that the endpoint of a segment which has an arrow is the terminal point of the segment. Notice that if A, B, C, D are not collinear, then $\overrightarrow{AB} \doteq \overrightarrow{CD}$ if and only if ABDC is a parallelogram. We need the fact that if $\overrightarrow{AB}$ is any directed line segment and if C is any point, then there is one and only one point D such that $\overrightarrow{AB} \doteq \overrightarrow{CD}$. We do not prove this fact, but assume that it is known from the study of geometry.

Definition 11-1b: Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be any two directed line segments. Then by their sum $\overrightarrow{AB} + \overrightarrow{CD}$ we mean the directed line segment $\overrightarrow{AX}$, where X is the unique point such that $\overrightarrow{BX} \doteq \overrightarrow{CD}$. We call the operation which assigns their sum to each pair of directed line segments the addition operation for directed line segments. Figure 11-1c shows some sums of directed line segments.



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{CD}$

$\overrightarrow{CX} = \overrightarrow{CD} + \overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{AB} + \overrightarrow{CD}$

Fig. 11-1c

Directed line segments can be added and multiplied by real numbers in a useful way. We give the formal definition of these operations here. Their properties are studied and applied throughout the rest of the chapter.

Definition: Let $\overrightarrow{AB}$ be any directed line segment and let r be any real number. Then the product $r\overrightarrow{AB}$ is the directed line segment $\overrightarrow{AX}$, where X is determined as follows:

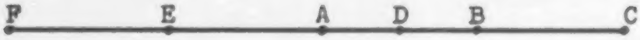
(1) If $r > 0$, then X is on the ray $\overrightarrow{AB}$ and $|AX| = r|AB|$.

(2) If $r < 0$, then X is on the ray opposite to $\overrightarrow{AB}$ and $|AX| = -r|AB|$.

(3) If $r = 0$, then $X = A$.

(4) If $B = A$, then $X = A$.

Figure 11-1d shows some typical products.



$$0\,\overrightarrow{AB} = \overrightarrow{AA}$$

$$1\,\overrightarrow{AB} = \overrightarrow{AB}$$

$$2\,\overrightarrow{AB} = \overrightarrow{AC}$$

$$\frac{1}{2}\,\overrightarrow{AB} = \overrightarrow{AD}$$

$$-1\,\overrightarrow{AB} = \overrightarrow{AE}$$

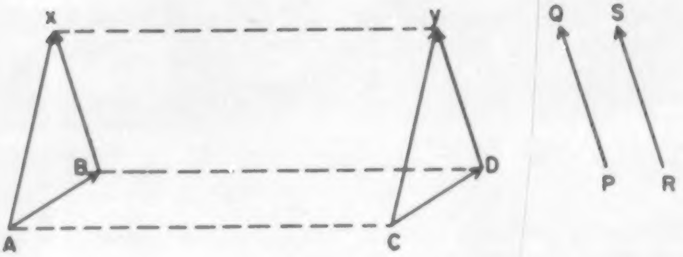
$$-2\,\overrightarrow{AB} = \overrightarrow{AF}$$

Fig. 11-1d

It is useful to know that if equivalent directed line segments are added to equivalent directed line segments the sums are equivalent, and that if equivalent directed line segments are multiplied by the same number the products are equivalent. We now state these facts formally as theorems and illustrate them.

Theorem 11-1a: If $\overrightarrow{AB} = \overrightarrow{CD}$ and if $\overrightarrow{PQ} = \overrightarrow{RS}$ then

$$\overrightarrow{AB} + \overrightarrow{PQ} = \overrightarrow{CD} + \overrightarrow{RS} .$$



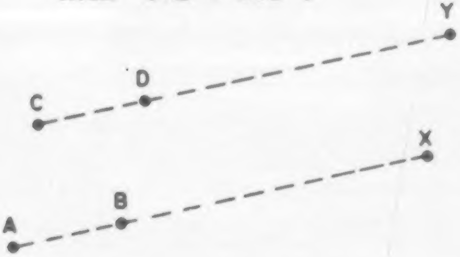
$\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{PQ}$, $\overrightarrow{CY} = \overrightarrow{CD} + \overrightarrow{RS}$

$$\overrightarrow{AX} \doteq \overrightarrow{CY}.$$

Fig. 11-1e

Figure 11-1e shows a typical instance of this theorem. It is equivalent to the fact that if ABDC is a parallelogram and if XYDB is a parallelogram, then AXYC is a parallelogram. This is a special case of a famous theorem of geometry known as Desargues' Theorem.

Theorem 11-1b: If $\overrightarrow{AB} \doteq \overrightarrow{CD}$ and if r is any real number, then $r\overrightarrow{AB} \doteq r\overrightarrow{CD}$.



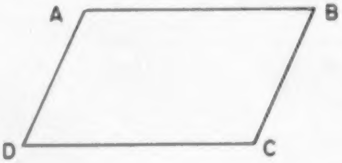
$\overrightarrow{AX} = r\overrightarrow{AB}$; $\overrightarrow{CY} = r\overrightarrow{CD}$; $\overrightarrow{AX} \cdot \overrightarrow{CY}$.

Fig. 11-1f

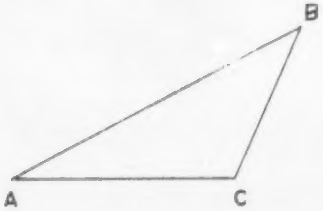
Figure 11-1f illustrates a case in which A, B, C, D are not collinear. It also illustrates the geometric version of the statement, that if ABDC is a parallelogram and if $\overrightarrow{AX} \doteq \overrightarrow{CY}$ then AXYC is a parallelogram.

Exercises 11-1

1. A and B are distinct points. List all the directed line segments they determine.
2. A, B and C are distinct points. List all the directed line segments they determine.
3. A, B, C and D are vertices of a parallelogram. List all the directed line segments they determine, and indicate which pairs are equivalent.



4. In triangle ABC
   - (a) $\overrightarrow{AB} + \overrightarrow{BC} = ?$
   - (b) $\overrightarrow{BA} + ? = \overrightarrow{BC}$.
   - (c) $? + \overrightarrow{BA} = \overrightarrow{BC}$.
   - (d) $? + \overrightarrow{AB} = \overrightarrow{AA}$.
   - (e) $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA} = ?$
   - (f) $\overrightarrow{BA} + (\overrightarrow{AC} + \overrightarrow{CB}) = ?$
   - (g) $? + \overrightarrow{AC} = \overrightarrow{CB}$.



5. A, B and X are collinear points. Find r such that

$$\overrightarrow{AX} = r\overrightarrow{AB}$$

and s such that

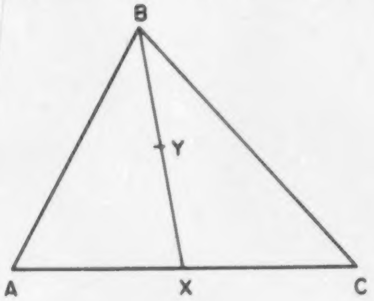
$$\overrightarrow{BX} = s\overrightarrow{BA} ,$$

if

(a) X is the midpoint of segment $\overline{AB}$.
(b) B is the midpoint of segment $\overline{AX}$.
(c) A is the midpoint of segment $\overline{BX}$.
(d) X is two-thirds of the way from A to B .
(e) B is two-thirds of the way from A to X .
(f) A is two-thirds of the way from B to X .

6. In triangle ABC , X is the midpoint of $\overline{AC}$ and Y is the midpoint of segment $\overline{BX}$.

(a) $\overrightarrow{BX} = \overrightarrow{BA} + ?\overrightarrow{AC}$.
(b) $\overrightarrow{BX} = ?\overrightarrow{BY}$.
(c) $\overrightarrow{BX} = \overrightarrow{BC} + ?$.
(d) $\overrightarrow{BX} = \overrightarrow{BC} + \frac{1}{2}\ ?$.
(e) $\overrightarrow{BY} = ?\overrightarrow{BX}$.
(f) $\overrightarrow{BY} = ?(\overrightarrow{BA} + \overrightarrow{AX})$.
(g) $\overrightarrow{BC} = ?\overrightarrow{BY} + \overrightarrow{XC}$.



11-2. Applications to Geometry.

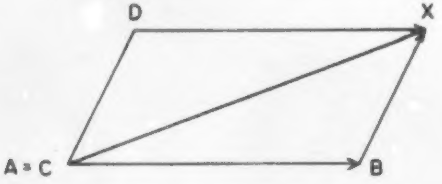
It is possible to use directed line segments to prove theorems of geometry. These proofs are based on algebraic properties of directed line segments. They are quite different from proofs usually given in geometry which appeal to such matters as congruent triangles and the like.

We state and illustrate the necessary algebraic properties of directed line segments here. We prove these statements in Section 11-3.

I. Commutative Law:

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{CD} + \overrightarrow{AB} .$$

Figure 11-2a shows an instance of the commutative law for addition in which the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have a common initial point.



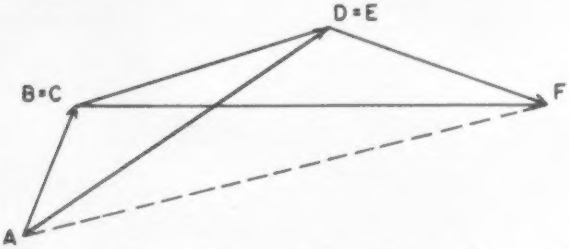
$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{CD} + \overrightarrow{AB} = \overrightarrow{AX} \quad .$$

Fig. 11-2a

II. Associative Law:

$$\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}) = (\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} \; .$$

Figure 11-2b shows sums $\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF})$ in which B and C are the same and D and E are the same.



$$\overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF}) = (\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} = \overrightarrow{AF} \quad .$$

Fig. 11-2b

III. Existence of Zero Elements.

Every directed line segment of the type $\overrightarrow{AA}$ is a zero element because $\overrightarrow{PQ} + \overrightarrow{AA} = \overrightarrow{PQ}$.

IV. Existence of Additive Inverses.

$\overrightarrow{BA}$ is the additive inverse of $\overrightarrow{AB}$, because $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA}$.

We use a minus sign to denote the additive inverse of a directed line segment $\overrightarrow{AB}$, and write $-\overrightarrow{AB}$ for $\overrightarrow{BA}$. We write $\overrightarrow{PQ} - \overrightarrow{AB}$ for $\overrightarrow{PQ} + \overrightarrow{BA}$.

This operation of subtraction is illustrated in Figure 11-2c.

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$$

$$\overrightarrow{AC} + \overrightarrow{BA} = \overrightarrow{BC}$$

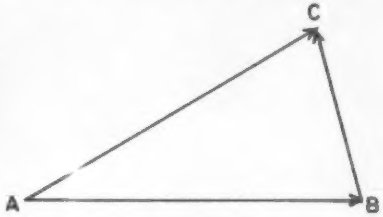
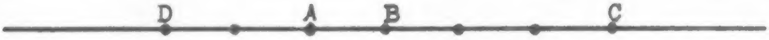


Fig. 11-2c

V. The Associative Law.

$r(s\overrightarrow{AB}) = (rs)\overrightarrow{AB}$.



$$-\tfrac{1}{2}(4\ \overrightarrow{AB}) = -\tfrac{1}{2}(\overrightarrow{AC}) = \overrightarrow{AD}$$

$$(-\tfrac{1}{2} \cdot 4)\overrightarrow{AB} = -\ 2\overrightarrow{AB} = \overrightarrow{AD}$$

Fig. 11-2d

Figure 11-2d shows an instance of the associative law in which $r = -\frac{1}{2}$, $s = 4$.

VI. **The Distributive Laws:**

$$r(\overrightarrow{AB} + \overrightarrow{CD}) = r\overrightarrow{AB} + r\overrightarrow{CD},$$

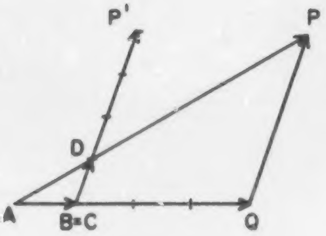
$$(r + s)\overrightarrow{AB} = r\overrightarrow{AB} + s\overrightarrow{AB} .$$

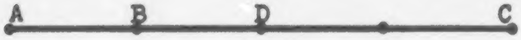
$$\overrightarrow{AQ} = 4\overrightarrow{AB}, \quad \overrightarrow{QP} = 4\overrightarrow{CD}, \quad \overrightarrow{AP} = 4\overrightarrow{AD}$$

$$\overrightarrow{AP} = \overrightarrow{AQ} + \overrightarrow{QP}$$

$$4\overrightarrow{AD} = 4\overrightarrow{AB} + 4\overrightarrow{CD}$$

$$4(\overrightarrow{AB} + \overrightarrow{CD}) = 4\overrightarrow{AB} + 4\overrightarrow{CD}$$





$$\overrightarrow{AD} = \overrightarrow{AC} - \overrightarrow{DC}$$

$$2\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CD}$$

$$(4 + (-2))\overrightarrow{AB} = 4\overrightarrow{AB} + (-2)\overrightarrow{AB}$$

Fig. 11-2e

Figure 11-2e illustrates the distributive laws for $r = 4$, $s = -2$.

The combined effect of all these laws can be summed up briefly as follows:

> Directed line segments obey the familiar rules of algebra with respect to addition, subtraction, and multiplication by real numbers.

We now show how this algebra of directed line segments can be applied to proving theorems of geometry.

Example 11-2a: Show that the midpoints of the sides of any quadrilateral are vertices of a parallelogram.

Proof: Let ABCD be the quadrilateral (see Figure 11-2f)

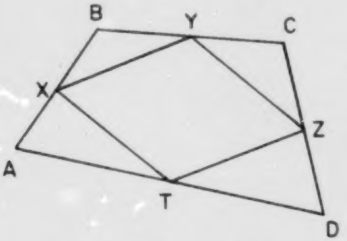


Fig. 11-2f

and let X, Y, Z, T be the midpoints of its sides as indicated. It is sufficient to show that $\overrightarrow{XY} \doteq \overrightarrow{TZ}$ since this implies both that $\overleftrightarrow{XY} \parallel \overleftrightarrow{TZ}$ and that $|XY| = |TZ|$.

We have
$$\overrightarrow{XY} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}$$
and
$$\overrightarrow{TZ} = \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{DC} .$$

Since $\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC}$, we also have
$$\begin{aligned}\overrightarrow{TZ} &\doteq \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}(\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC})\\ &\doteq \tfrac{1}{2}\overrightarrow{AD} - \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}\\ &\doteq \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC} .\end{aligned}$$

This shows that $\overrightarrow{XY} \doteq \overrightarrow{TZ}$.

Example 11-2b: Prove that the diagonals of a parallelogram bisect each other.

Solution: Let ABCD be the parallelogram (see Figure 11-2g).

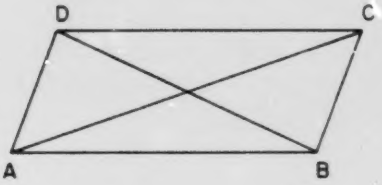


Fig. 11-2g

Then the midpoint of $\overline{AC}$ is the endpoint of $\frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$. The midpoint of $\overline{DB}$ is the endpoint of $\overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AD})$ which equals $\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$ or $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$. We show that this is the same as $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$. Since ABCD is a parallelogram $\overrightarrow{AD} \doteq \overrightarrow{BC}$, so the last sum is certainly equivalent to $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$. We conclude that these directed line segments are the same by noticing that in addition to being equivalent they also have the same initial point.

Example 11-2c: Prove that the medians of a triangle meet in a point which trisects each of them.

Solution: Let ABC be the triangle. (See Figure 11-2h.)

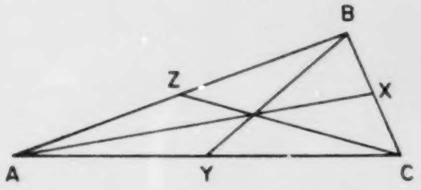


Fig. 11-2h

Let X, Y, Z be the midpoints of its sides. Then, the point two-thirds the way from A to X is the endpoint of $\frac{2}{3}(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC})$.

The point two-thirds the way from B to Y is the endpoint of

$$\overrightarrow{AB} + \frac{2}{3}(\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}) .$$

The point two-thirds the way from C to Z is the endpoint of

$$\overrightarrow{AC} + \frac{2}{3}(\overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}) .$$

We show that these three directed line segments are one and the same. We use the fact that $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$.

Then the first is equal to

$$\frac{2}{3}(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC})$$

which is equal to $\frac{2}{3}(\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC})$

or $\frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$.

The second is equal to $\overrightarrow{AB} - \frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}$ which also equals

$$\frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC} .$$

The third is equal to $\overrightarrow{AC} - \frac{2}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AB}$ which also equals

$$\frac{1}{3}\overrightarrow{AC} + \frac{1}{3}\overrightarrow{AB} .$$

Example 11-2d: Prove that the line which joins one vertex of a parallelogram to the midpoint of an opposite side is trisected by a diagonal. Prove also that it trisects this diagonal.

Solution: Let ABCD be the parallelogram (see Figure 11-21). Let A be the given vertex and let X be the midpoint of $\overline{CD}$.

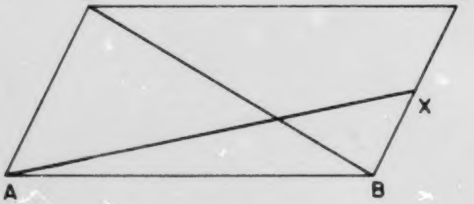


Fig. 11-21

We are to show that the point two-thirds of the way from A to X is the same as the point two-thirds of the way from D to B .

The first point is the endpoint of

$$\frac{2}{3}(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC})$$

or

$$\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{BC} .$$

The second point is the endpoint of

$$\overrightarrow{AD} + \frac{2}{3}(\overrightarrow{DA} + \overrightarrow{AB}) .$$

This latter equals

$$\overrightarrow{AD} - \frac{2}{3}\overrightarrow{AD} + \frac{2}{3}\overrightarrow{AB}$$

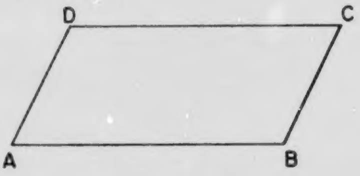
or

$$\frac{1}{3}\overrightarrow{AD} + \frac{2}{3}\overrightarrow{AB} .$$

Since $\overrightarrow{AD}$ is equivalent to $\overrightarrow{BC}$ we see that these two directed line segments are equivalent; that they are in fact the same follows from the additional fact that they have the same initial point.

Exercises 11-2

1. If ABCD is a parallelogram, express $\overrightarrow{DB}$.
   (a) in terms of $\overrightarrow{DC}$ and $\overrightarrow{DA}$.
   (b) in terms of $\overrightarrow{DC}$ and $\overrightarrow{CB}$.
   (c) in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
   (d) in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.
   (e) in terms of $\overrightarrow{BA}$ and $\overrightarrow{BC}$.



2. If A and B are distinct points, identify the set of all terminal points of the directed line segments of the form $t\,\overrightarrow{AB}$ for which
   (a) $t \geq 0$.
   (b) $0 \leq t \leq 1$.
   (c) $t \geq 1$.
   (d) $-1 \leq t \leq 1$.

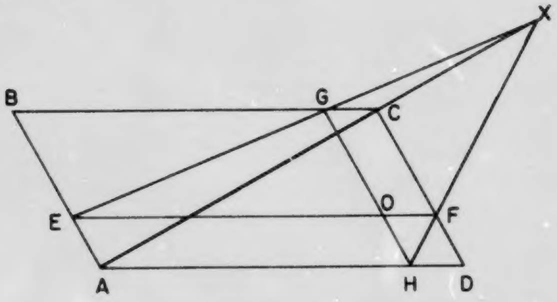
3. If A, B, C are non-collinear points, find the set of all terminal points of directed line segments of the form

$$r\ \overrightarrow{AB} + s\ \overrightarrow{AC}$$

for which
   - (a) $r = 0$, s arbitrary.
   - (b) $s = 0$, r arbitrary.
   - (c) $0 \le r \le 1$, s arbitrary.
   - (d) $0 \le s \le 1$, r arbitrary.
   - (e) $0 \le r \le 1$, $0 \le s \le 1$.
   - (f) $r = 1$, s arbitrary.
   - (g) $s = 1$, r arbitrary.
   - *(h) $r + s = 1$.
   - *(i) $r - s = 1$.
   - *(j) $\frac{r}{2} + \frac{s}{3} = 1$.
   - *(k) $6r + 7s = 8$
   - *(l) $ar + bs + c = 0$, where $\underline{a}$, $\underline{b}$, $\underline{c}$ are real numbers and where not both $\underline{a}$ and $\underline{b}$ are zero.

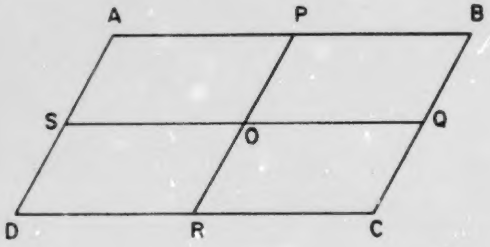
4. Show by an example that subtraction of directed line segments
   - (a) is not commutative,
   - (b) is not associative.

*5. In the following figure



ABCD, EOGB, and HDFO are each parallelograms. Prove that their respective diagonals $\overline{AC}$, $\overline{EG}$, $\overline{HF}$, extended if necessary, meet in a single point X .

6. ABCD is a parallelogram and P, Q, R, S are the midpoints of the sides.



For each of the following directed line segments, find an equivalent directed line segment of the form r $\overrightarrow{OQ}$ + s $\overrightarrow{OP}$.

(a) $\overrightarrow{OB}$ (e) $\overrightarrow{DB}$
(b) $\overrightarrow{OC}$ (f) $\overrightarrow{AC}$
(c) $\overrightarrow{OD}$ (g) $\overrightarrow{CA}$
(d) $\overrightarrow{OA}$ (h) $\overrightarrow{BD}$

7. Show that the four diagonals of a parallelepiped bisect one another.

11-3. Vectors and Scalars; Components.

Directed line segments acquire new properties when algebraic operations are defined for them, so it is proper to give them a new name. Real numbers also acquire new properties when they multiply directed line segments, so it is proper to rename them also. From now on we shall call a directed line segment a vector. We shall call a real number a scalar if and when it multiplies a vector. This is a refinement which is not absolutely necessary for logical thinking, but it helps.

We are going to discuss equivalence of vectors, addition of vectors and multiplication of vectors by scalars in terms of coordinates. The following theorem is the basic tool in this discussion.

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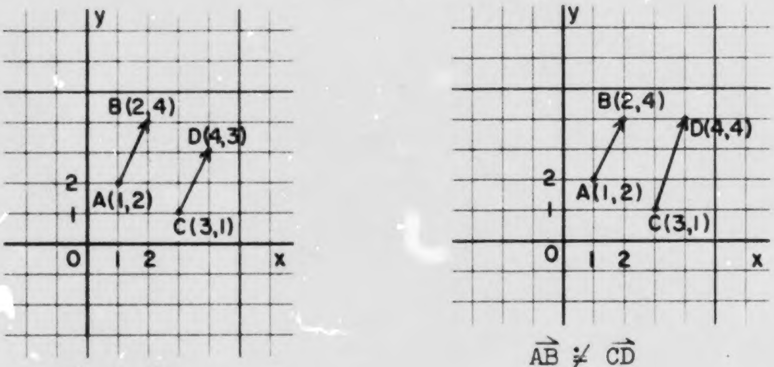
Theorem 11-3a: Let A, B, C, D have respective coordinates (a_1,a_2), (b_1,b_2), (c_1,c_2), (d_1,d_2). Then

$$\overrightarrow{AB} \doteq \overrightarrow{CD}$$

if and only if

$$b_1 - a_1 = d_1 - c_1 \text{ and } b_2 - a_2 = d_2 - c_2 .$$

Proof: Figure 11-3a illustrates Theorem 11-3a.



$\overrightarrow{AB} \doteq \overrightarrow{CD}$

$2 - 1 = 4 - 3$
$4 - 2 = 3 - 1$

$\overrightarrow{AB} \not\doteq \overrightarrow{CD}$

$4 - 2 \neq 4 - 1$

Fig. 11-3a

We give only a few indications of its proof.

If $b_1 - a_1 = d_1 - c_1$ and if $b_2 - a_2 = d_2 - c_2$ then

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 = (d_1 - c_1)^2 + (d_2 - c_2)^2$$

and

$$\frac{b_2 - a_2}{b_1 - a_1} = \frac{d_2 - c_2}{b_2 - a_2}$$

provided that

$$b_1 - a_1 \neq 0 \text{ and } b_2 - a_2 \neq 0 .$$

We conclude that $|AB| = |CD|$ and that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. This makes plausible the fact that if the given equations hold then $\overrightarrow{AB} \doteq \overrightarrow{CD}$. It doesn't completely prove this (we need $\overrightarrow{AB} \parallel \overrightarrow{CD}$) and it doesn't contribute at all to the proof of the converse.

Corollary: If $\overrightarrow{OP}$ is the vector equivalent to $\overrightarrow{AB}$, where O is the origin, then P has coordinates $(b_1 - a_1, b_2 - a_2)$.

Definition 11-3a: If A is the point (a_1,a_2) and B is the point (b_1,b_2), we call the number $b_1 - a_1$ the x-component of $\overrightarrow{AB}$, the number $b_2 - a_2$ the y-component of $\overrightarrow{AB}$.

In most discussions of vectors the initial and terminal points of the vectors are not as important as their x and y-components. We shall therefore often specify a vector by giving its x and y component. We use square brackets [,] to do this; $[p,q]$ means any vector whose x-component is p and whose y-component is q. We shall sometimes denote vectors by single letters, with an arrow above, like $\overrightarrow{A}$, when the specific endpoints are not important. We also write $\overrightarrow{A} = \overrightarrow{B}$ to assert that two vectors are equivalent. The equal sign should properly connect not the vectors themselves but their components. Thus Theorem 11-3a can be restated as follows:

If $\overrightarrow{X}$ is $[x_1,x_2]$ and $\overrightarrow{Y}$ is $[y_1,y_2]$, then

$$\overrightarrow{X} = \overrightarrow{Y}$$

if and only if

$$x_1 = y_1 \quad \text{and} \quad x_2 = y_2 .$$

We use the symbol $|X|$ to denote the length of X. We have

$$|[x_1,x_2]| = \sqrt{x_1^2 + x_2^2} .$$

We turn now to the addition and multiplication operations for vectors, show how they can be effected in terms of components and prove the basic algebraic laws stated for them in Section 11-2.

__Theorem__ __11-3b__: If $\overrightarrow{X}$ is $[x_1, x_2]$ and if $\overrightarrow{Y}$ is $[y_1, y_2]$ then $\overrightarrow{X} + \overrightarrow{Y}$ is $[(x_1 + y_1), (x_2 + y_2)]$.

__Proof__: By definition of addition for vectors (see Figure 11-3b)

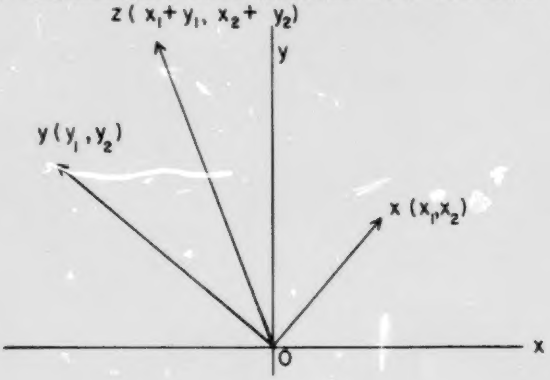


Fig. 11-3b

$\overrightarrow{OZ}$ is $\overrightarrow{OX} + \overrightarrow{OY}$ if and only if $\overrightarrow{XZ} \doteq \overrightarrow{OY}$. According to Theorem 11-3a, this will be so if and only if the point Z is $(x_1 + y_1 , x_2 + y_2)$. It follows that the components of $\overrightarrow{X} + \overrightarrow{Y}$ are $x_1 + y_1$ and $x_2 + y_2$.

__Corollary__: Addition of vectors is commutative.

$$\overrightarrow{X} + \overrightarrow{Y} = \overrightarrow{Y} + \overrightarrow{X} .$$

__Corollary__: Addition of vectors is associative.

$$(\overrightarrow{X} + \overrightarrow{Y}) + \overrightarrow{Z} = \overrightarrow{X} + (\overrightarrow{Y} + \overrightarrow{Z}) .$$

Corollary: There is a zero vector $[0,0]$.

Corollary: Every vector $\overrightarrow{X}$ has an additive inverse $-\overrightarrow{X}$. If $\overrightarrow{X}$ is $[x_1,x_2]$, then $-\overrightarrow{X}$ is $[-x_1,-x_2]$.

Theorem 11-3c: If $\overrightarrow{X}$ is $[x_1,x_2]$, then $r\overrightarrow{X}$ is $[rx_1,rx_2]$.

Proof: Let Y be the point (rx_1,rx_2) (see Figure 11-3c).

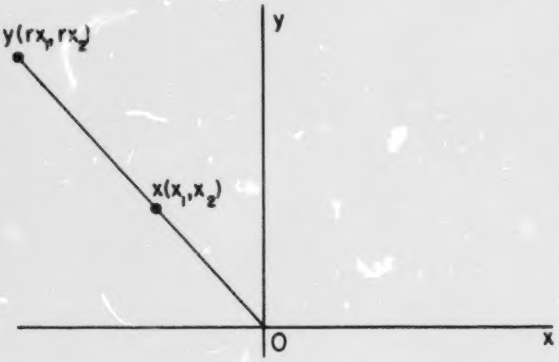


Fig. 11-3c

Then

$$|OY| = \sqrt{(rx_1)^2 + (rx_2)^2} = |r|\sqrt{x_1{}^2 + x_2{}^2}$$

$$= |r| \cdot |OX| .$$

Also O, X, Y are collinear, since they are on the line whose equation is $x_2x - x_1y = 0$. We must show that the ray $\overrightarrow{OX}$ is parallel to the ray $\overrightarrow{OY}$ to complete our proof. We omit this part of the proof.

Corollary: Multiplication by scalars is associative.

$$r(s\overrightarrow{X}) = (rs)\overrightarrow{X} .$$

Corollary: Multiplication by scalars obeys the distributive laws.

$$r(\overrightarrow{X} + \overrightarrow{Y}) = r\overrightarrow{X} + r\overrightarrow{Y}$$

$$(r + s)\overrightarrow{X} = r\overrightarrow{X} + r\overrightarrow{X} .$$

Corollary: $(-1)\overrightarrow{X} = \overrightarrow{-X}$.

Corollary: If $\overrightarrow{X}$ is $[x_1,x_2]$ and if $\overrightarrow{Y}$ is $[y_1,y_2]$ then $r\overrightarrow{X} + s\overrightarrow{Y}$ is $[rx_1 + sy_1 , rx_2 + sy_2]$.

Definition 11-3b: Non zero vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ are said to be parallel if and only if the directed line segments $\overrightarrow{OX}$ and $\overrightarrow{OY}$ equivalent to them are collinear.

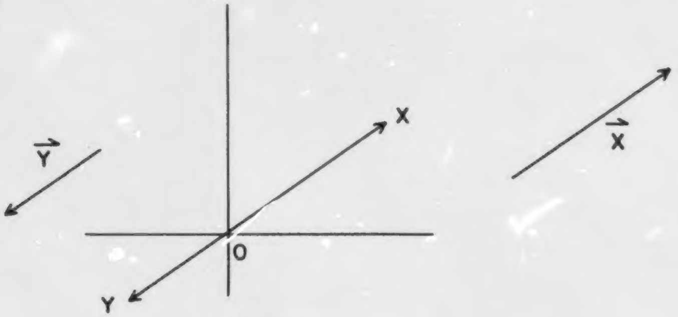


Fig. 11-3c

Theorem 11-3d: Non zero vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ are parallel if and only if

$$\overrightarrow{Y} = r\overrightarrow{X}$$

for some non-zero real number r .

Proof: Let $\overrightarrow{X}$ be $[x_1,x_2]$ and $\overrightarrow{Y}$ be $[y_1,y_2]$, let X be the point (x_1,x_2) and Y be the point (y_1,y_2) .

Then $\overrightarrow{OX} \doteq \overrightarrow{X}$, $\overrightarrow{OY} \doteq \overrightarrow{Y}$. Then $\overrightarrow{X} \parallel \overrightarrow{Y}$ if and only if O, X and Y are collinear. But

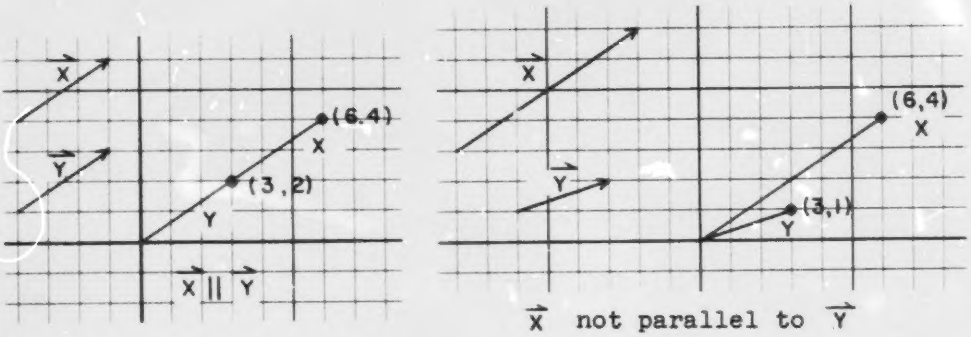
$$\overrightarrow{OX} = r\,\overrightarrow{OY}$$

if and only if

$$x_1 = ry_1$$

$$x_2 = ry_2$$

which holds if and only if O, X, Y are collinear.


$\overrightarrow{X}$
$\overrightarrow{Y}$
(6,4)
X
(3,2)
Y
$\overrightarrow{X} \parallel \overrightarrow{Y}$
$\overrightarrow{X}$
(6,4)
X
$\overrightarrow{Y}$
(3,1)
Y

$\overrightarrow{X}$ not parallel to $\overrightarrow{Y}$

Fig. 11-3d

Theorem 11-3e: Let $\overrightarrow{X}$ and $\overrightarrow{Y}$ be any pair of non-zero, non-parallel vectors. Then for each vector $\overrightarrow{Z}$ there are numbers r and s such that

$$\overrightarrow{Z} = r\overrightarrow{X} + s\overrightarrow{Y} .$$

Proof: Let $\overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{Z}$ be $[x_1,x_2]$, $[y_1,y_2]$, $[z_1,z_2]$. We are to show that the equations for r , s

$$z_1 = rx_1 + sy_1$$

$$z_2 = rx_2 + sy_2$$

have a unique solution (r,s) . Since $\overrightarrow{X}$ is not parallel to $\overrightarrow{Y}$, it follows from Theorem 11-3d that

$$x_1y_2 - y_1x_2 \neq 0 .$$

Our conclusion now follows from the result of Chapter 7, Section 3 on the existence and uniqueness of solution of equations.

Corollary: If $r\overrightarrow{X} + s\overrightarrow{Y} = \overrightarrow{0}$ (where $\overrightarrow{0}$ is a zero vector) then $r = s = 0$.

Definition 11-3c: Any two non-zero, non-parallel vectors in the plane are said to be a base for all the vectors of the plane.

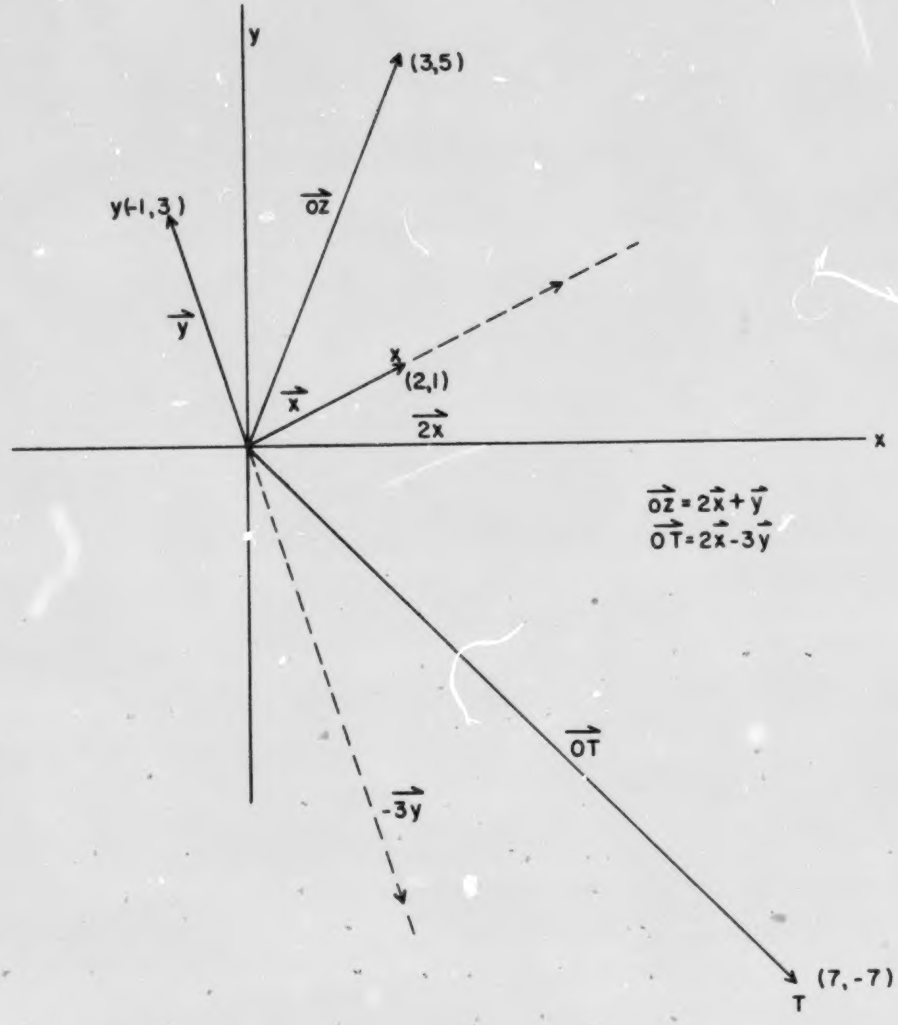

y
(3,5)
y(-1,3)
$\overrightarrow{OZ}$
$\vec{y}$
x
(2,1)
$\vec{x}$
$\overrightarrow{2x}$
x
$\overrightarrow{OZ} = 2\vec{x} + \vec{y}$
$\overrightarrow{OT} = 2\vec{x} - 3\vec{y}$
$\overrightarrow{OT}$
$-\overrightarrow{3y}$
(7, -7)
T

Fig. 11-3e

Figure 11-3e shows two base vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ and vectors $\overrightarrow{OZ}$ and $\overrightarrow{OT}$ expressed in the form $r\overrightarrow{X} + s\overrightarrow{Y}$.

[1,0] and [0,1] form a base which is frequently used. The vector [1,0] is denoted by $\overrightarrow{i}$ and the vector [0,1] is denoted by $\overrightarrow{j}$.

Theorem 11-3f: $\overrightarrow{X} = a\overrightarrow{i} + b\overrightarrow{j}$ if and only if $\overrightarrow{X}$ is [a,b] and (a,b) is the point P for which

$$\overrightarrow{X} = \overrightarrow{OP} .$$

Proof: If $\overrightarrow{X}$ is [a,b] , then, since

$$[a,b] = a[1,0] + b[0,1]$$

it follows that

$$\overrightarrow{X} = a\overrightarrow{i} + b\overrightarrow{j} .$$

If $\overrightarrow{X} = a\overrightarrow{i} + b\overrightarrow{j}$, then

$$\overrightarrow{X} = a[1,0] + b[0,1] = [a,b] .$$

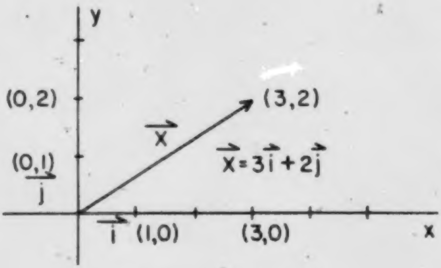


Fig. 11-3f

Figure 11-3f shows an example of a vector $\overrightarrow{X}$ expressed as a sum $3\overrightarrow{i} + 2\overrightarrow{j}$.

Exercises 11-3

1. If A, B and C are respectively $(1,2)$, $(4,3)$, $(6,1)$ find X so that
   (a) $\overrightarrow{AB} \doteq \overrightarrow{CX}$.
   (b) $\overrightarrow{AX} \doteq \overrightarrow{CB}$.
   (c) $\overrightarrow{XA} \doteq \overrightarrow{CB}$.
   (d) $\overrightarrow{XA} \doteq \overrightarrow{BC}$.
2. Same as Problem 1, if A, B, C are respectively $(-1,2)$, $(4,-3)$, $(-6,-1)$.
3. Find the components of
   (a) $[3,2] + [4,1]$.
   (b) $[3,-2] + [-4,1]$.
   (c) $4[5,6]$.
   (d) $-4[5,6]$.
   (e) $-1[5,6]$.
   (f) $-[5,6]$.
   (g) $3[4,1] + 2[-1,3]$.
   (h) $3[4,1] - 2[-1,3]$.
4. Determine x and y so that
   (a) $x[3,-1] + y[3,1] = [5,6]$.
   (b) $x[3,2] + y[2,3] = [1,2]$.
   (c) $x[3,2] + y[-2,3] = [5,6]$.
   (d) $x[3,2] + y[6,4] = [-3,-2]$ (Infinitely many solutions).
5. Determine a and b so that
   (a) $[3,1] = a\vec{i} + b\vec{j}$.
   (b) $[1,-3] = a\vec{i} + b\vec{j}$.
   (c) $\vec{i} = a[-3,1] + b[1,-3]$.
   (d) $\vec{j} = a[-3,1] + b[1,-3]$.
6. Determine a and b so that $3\vec{i} - 2\vec{j} = a(3\vec{i} + 4\vec{j}) + b(4\vec{i} + 3\vec{j})$.

11-4. Inner Product.

Our algebra of vectors does not yet include multiplication of one vector by another. We now define such a product.

We first say what we mean by the angle between two vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ which do not necessarily have a common initial point.

Definition 11-4a: Let $\overrightarrow{X}$ and $\overrightarrow{Y}$ be any non-zero vectors and let $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be vectors whose initial point is the origin O and which are equivalent respectively to $\overrightarrow{X}$ and $\overrightarrow{Y}$. Then by the angle between $\overrightarrow{X}$ and $\overrightarrow{Y}$ we mean the angle between $\overrightarrow{OX}$ and $\overrightarrow{OY}$.

Definition 11-4b: Let $\overrightarrow{X}$ and $\overrightarrow{Y}$ be any vectors. Then the inner product of $\overrightarrow{X}$ and $\overrightarrow{Y}$ is the real number

$$|\overrightarrow{X}|\ |\overrightarrow{Y}| \cos \theta$$

where $|\overrightarrow{X}|$ is the length of $\overrightarrow{X}$, $|\overrightarrow{Y}|$ is the length of $\overrightarrow{Y}$ and θ is the angle between $\overrightarrow{X}$ and $\overrightarrow{Y}$. (If $\overrightarrow{X}$ or $\overrightarrow{Y}$ is a zero vector then θ is not defined. We interpret the definitions to mean that the inner product is zero, in this case.)

The inner product has important properties. Before we investigate these properties of the inner product we relate the inner product to a familiar mathematical relation--the law of cosines.

If our given vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ are not parallel they determine a triangle OXY, where O is the origin and where X and Y are endpoints of the vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$ respectively equivalent to $\overrightarrow{X}$ and $\overrightarrow{Y}$. We can find at least one earlier appearance of the inner product by applying the law of cosines to the triangle. It asserts (Figure 11-4a)

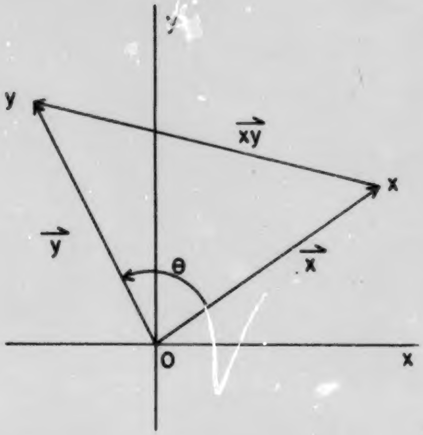


Fig. 11-4a

$$|\overrightarrow{XY}|^2 = |\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2 - 2|\overrightarrow{OX}| \cdot |\overrightarrow{OY}| \cos \theta$$

so that

$$|\overrightarrow{OX}| \cdot |\overrightarrow{OY}| \cos \theta = \frac{|\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2 - |\overrightarrow{XY}|^2}{2} .$$

Thus the expression we have called the "inner product" is suggested by the law of cosines.

We sometimes denote this product by the symbols $\overrightarrow{X} \cdot \overrightarrow{Y}$ (read "$\overrightarrow{X}$ dot $\overrightarrow{Y}$") and sometimes call it the "dot product."

Usually, in algebra, a multiplication operation for a set of objects assigns a member of this set to each pair of its members. The inner product is not an operation of this type. It does not assign a vector to a pair of vectors but rather it assigns a real number to each pair of vectors.

Example 11-4a: Evaluate $\overrightarrow{X} \cdot \overrightarrow{Y}$ if $|\overrightarrow{X}| = 2$, $|\overrightarrow{Y}| = 3$ and (a) $\theta = 0$, (b) $\theta = 45^\circ$, (c) $\theta = 90^\circ$, (d) $\theta = 180^\circ$.

Solution:

$$\begin{aligned}
&(a)\quad \overrightarrow{X} \cdot \overrightarrow{Y} = 2 \cdot 3 \cos 0^\circ = 2 \cdot 3 \cdot 1 = 6\\
&(b)\quad \overrightarrow{X} \cdot \overrightarrow{Y} = 2 \cdot 3 \cos 45^\circ = 2 \cdot 3 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}\\
&(c)\quad \overrightarrow{X} \cdot \overrightarrow{Y} = 2 \cdot 3 \cos 90^\circ = 2 \cdot 3 \cdot 0 = 0\\
&(d)\quad \overrightarrow{X} \cdot \overrightarrow{Y} = 2 \cdot 3 \cos 180^\circ = 2 \cdot 3 \cdot (-1) = -6
\end{aligned}$$

The inner product has many applications. One of these is a test for perpendicularity.

Theorem 11-4a: If $\overrightarrow{X}$ and $\overrightarrow{Y}$ are non-zero vectors, then they are perpendicular if and only if

$$\overrightarrow{X} \cdot \overrightarrow{Y} = 0 .$$

Proof: According to the definition of inner product

$$\overrightarrow{X} \cdot \overrightarrow{Y} = |\overrightarrow{X}| \cdot |\overrightarrow{Y}| \cos \theta \quad .$$

This product of real numbers is zero if and only if one of its factors is zero. Since $\overrightarrow{X}$ and $\overrightarrow{Y}$ are non-zero vectors, the numbers $|\overrightarrow{X}|$ and $|\overrightarrow{Y}|$ are not zero. Therefore the product is zero if and only if $\cos \theta = 0$, which is the case if and only if $\overrightarrow{X}$ and $\overrightarrow{Y}$ are perpendicular.

The following theorem supplies a useful formula for the inner product of vectors.

Theorem 11-4b: If $\overrightarrow{X} = [x_1, x_2]$, $\overrightarrow{Y} = [y_1, y_2]$ then

$$\overrightarrow{X} \cdot \overrightarrow{Y} = x_1 y_1 + x_2 y_2 .$$

Proof: According to the law of cosines (see Figure 11-4b)

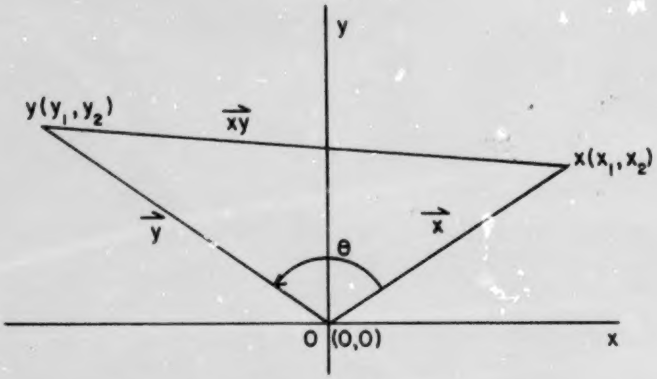

y
$y(y_1,y_2)$
$\overrightarrow{xy}$
$x(x_1,x_2)$
$\vec{y}$
$\vec{x}$
θ
O
(0,0)
x

Fig. 11-4b

$$|\overrightarrow{OX}| \cdot |\overrightarrow{OY}| \cos \theta = \frac{|\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2 - |\overrightarrow{XY}|^2}{2}$$

$$= \frac{x_1{}^2 + x_2{}^2 + y_1{}^2 + y_2{}^2 - (x_1-y_1)^2 - (x_2-y_2)^2}{2}$$

$$= x_1y_1 + x_2y_2 \ .$$

ice, by definition, the left member of this equation is $\vec{X} \cdot \vec{Y}$, theorem is proved.

Example 11-4b: If $\vec{X}$ is [3,4] and $\vec{Y}$ is [5,2], find $\vec{Y}$.

Solution: $\vec{X} \cdot \vec{Y} = 3 \cdot 5 + 4 \cdot 2$

$= 23$.

Example 11-4c: If $\overrightarrow{X}$ is $[3,7]$ and $\overrightarrow{Y}$ is $[-7,3]$, show that $\overrightarrow{X}$ and $\overrightarrow{Y}$ are perpendicular.

Solution: $\overrightarrow{X} \cdot \overrightarrow{Y} = 3(-7) + 7 \cdot 3 = 0$.

The conclusion follows from Theorem 11-4a, and the fact that $\overrightarrow{X}$ and $\overrightarrow{Y}$ are non-zero.

A useful fact about inner products is that they have some of the algebraic properties of products of numbers. The following theorem gives one such common property.

Theorem 11-4c: If $\overrightarrow{X}$, $\overrightarrow{Y}$, $\overrightarrow{Z}$ are any vectors, then

$$\overrightarrow{X} \cdot (\overrightarrow{Y} + \overrightarrow{Z}) = \overrightarrow{X} \cdot \overrightarrow{Y} + \overrightarrow{X} \cdot \overrightarrow{Z}$$

$$(t\overrightarrow{X}) \cdot \overrightarrow{Y} = t(\overrightarrow{X} \cdot \overrightarrow{Y}) .$$

Proof: Let $\overrightarrow{X} = [x_1,x_2]$, $\overrightarrow{Y} = [y_1,y_2]$, $\overrightarrow{Z} = [z_1,z_2]$. Then

$$\begin{aligned}
\overrightarrow{X} \cdot (\overrightarrow{X} + \overrightarrow{Z}) &= [x_1,x_2] \cdot [y_1 + z_1 , y_2 + z_2] \\
&= x_1(y_1 + z_1) + x_2(y_2 + z_2) \\
&= x_1y_1 + x_2y_2 + x_1z_1 + x_2z_2 \\
&= \overrightarrow{X} \cdot \overrightarrow{Y} + \overrightarrow{X} \cdot \overrightarrow{Z}
\end{aligned}$$

$$\begin{aligned}
(t\overrightarrow{X}) \cdot \overrightarrow{Y} &= [tx_1,tx_2] \cdot [y_1,y_2] \\
&= tx_1y_1 + tx_2y_2 \\
&= t(x_1y_1 + x_2y_2) \\
&= t(\overrightarrow{X} \cdot \overrightarrow{Y}) .
\end{aligned}$$

Corollary: $\vec{X} \cdot (a\vec{Y} + b\vec{Z}) = a(\vec{X} \cdot \vec{Y}) + b(\vec{X} \cdot \vec{Z})$.

In certain applications of vectors to physics the notion of a component of a vector in the direction of another vector is important. We now define this concept.

Definition 11-4c: Let $\vec{X}$ be any non-zero vector and let $\vec{Y}$ be any vector. Then the component of $\vec{Y}$ in the direction of $\vec{X}$ is the number given by each of the following equal expressions:

$$\frac{\vec{X} \cdot \vec{Y}}{|\vec{X}|} = \frac{|\vec{X}| \cdot |\vec{Y}| \cos \theta}{|\vec{X}|} = |\vec{Y}| \cos \theta .$$

NOTE: The component of $\vec{Y}$ in the direction of $\vec{X}$ can be described geometrically (see Figure 11-4c).

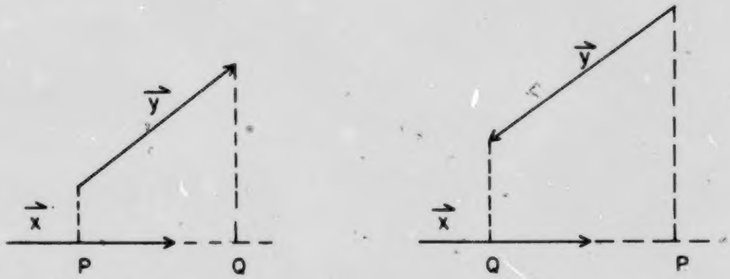


Fig. 11-4c

In both parts of the figure P is the foot of the perpendicular from the initial point of $\vec{Y}$ to the line of $\vec{X}$, and Q is the foot of the perpendicular from the terminal point of $\vec{Y}$ to this line. In the first part the component of $\vec{Y}$ in the direction of $\vec{X}$ turns out to be the distance from P to Q . In the second part this component turns out to be the negative of the distance from P to Q .

The inner product is used frequently in applications of vectors to physics. For the moment we consider inner products from a purely mathematical standpoint.

Example 11-4d: Let $\overrightarrow{X}$ be any vector parallel to the positive x-axis, let $\overrightarrow{Y}$ be any vector parallel to the positive y-axis and let $\overrightarrow{Z}$ be the vector $[p,q]$. Show that p and q are the components of $\overrightarrow{Z}$ in the direction of $\overrightarrow{X}$ and $\overrightarrow{Y}$ respectively.

Solution: According to Theorem 10-5a

$$\cos\theta = \frac{p}{\sqrt{p^2 + q^2}}$$

so $$p = \cos\theta \cdot \sqrt{p^2 + q^2}\ .$$

Since $|Z| = \sqrt{p^2 + q^2}$, we conclude that

$$p = |\overrightarrow{Z}| \cos\theta\ .$$

The angle between $\overrightarrow{Z}$ and the y-axis is $\frac{\pi}{2} - \theta$. Consequently the component of $\overrightarrow{Z}$ in the direction of $\overrightarrow{Y}$ is

$$\cos(\tfrac{\pi}{2} - \theta)\sqrt{p^2 + q^2}\ .$$

Since $\cos(\frac{\pi}{2} - \theta) = \sin\theta$ and since $\sin\theta = \frac{q}{\sqrt{p^2 + q^2}}$, we conclude that this component is, in fact,

$$\frac{q}{\sqrt{p^2 + q^2}} \cdot \sqrt{p^2 + q^2}\ , \text{or}\ q\ .$$

Vectors in Three Dimensions: Much of our discussion of vectors in the plane can be carried over to three dimensions with only minor modifications.

The portions about directed line segments require no modification. When we come to coordinates and components, the conclusions are as follows:

1. The components of a vector in three dimensional space are an ordered triple $[a,b,c]$ of real numbers.
2. Two vectors $[a,b,c]$ and $[p,q,r]$ are equal if and only if $a = p$, $b = q$ and $c = r$.
3. The addition of vectors $[a,b,c]$ and $[p,q,r]$ is given by the rule
$$[a,b,c] + [p,q,r] = [a + p,\ b + q,\ c + r] .$$
4. Scalar multiplication of vectors is given by the rule
$$r[a,b,c] = [ra,rb,rc].$$
5. The unit base vectors associated with the coordinate axes are
$$\vec{i} = [1,0,0]$$
$$\vec{j} = [0,1,0]$$
$$\vec{k} = [0,0,1] .$$

Figure 11-4d shows these base vectors.

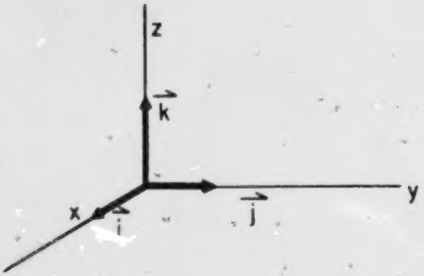


Fig. 11-4d

The vector $\vec{V} = 4\vec{i} + 8\vec{j} + 8\vec{k}$ is illustrated in Figure 11-4e.

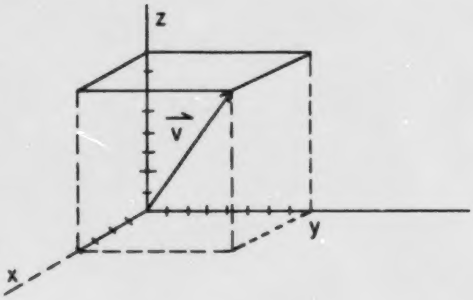


Fig. 11-4e

6. The inner product of $\vec{V}$ and $\vec{W}$ is still given by

$$\vec{V} \cdot \vec{W} = |\vec{V}||\vec{W}| \cos \theta \quad .$$

In component form if $\vec{V}$ is $[v_1,v_2,v_3]$ and $\vec{W}$ is $[w_1,w_2,w_3]$, then

$$\vec{V} \cdot \vec{W} = v_1w_1 + v_2w_2 + v_3w_3$$

also $$|\vec{V}| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad .$$

Exercises 11-4

1. Find $\vec{X} \cdot \vec{Y}$ if
   (a) $\vec{X} = \vec{i}$, $\vec{Y} = \vec{j}$.
   (b) $\vec{X} = \vec{i}$, $\vec{Y} = \vec{i}$.
   (c) $\vec{X} = \vec{j}$, $\vec{Y} = \vec{i}$.
   (d) $\vec{X} = \vec{j}$, $\vec{Y} = \vec{j}$.

(e) $\vec{X} = \vec{i} + \vec{j}$, $\vec{Y} = \vec{i} - \vec{j}$.

(f) $\vec{X} = 2\vec{i} + 3\vec{j}$, $\vec{Y} = 4\vec{i} - 5\vec{j}$.

(g) $\vec{X} = -2\vec{i} - 3\vec{j}$, $\vec{Y} = -4\vec{i} + 5\vec{j}$.

(h) $\vec{X} = a\vec{i} + b\vec{j}$, $\vec{Y} = c\vec{i} + d\vec{j}$.

(i) $\vec{X} = a\vec{i} + b\vec{j}$, $\vec{Y} = 4\vec{X}$.

(j) $\vec{X} = a\vec{i} + b\vec{j}$, $\vec{Y} = s\vec{X}$.

2. Find the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$ and $\vec{X} \cdot \vec{Y}$ is

   (a) 0 , (b) 1 , (c) -2 , (d) 3 , (e) -4 , (f) 5 , (g) 6 , (h) -6 .

3. If $\vec{X} = 3\vec{i} + 4\vec{j}$, determine a so that $\vec{Y}$ is perpendicular to $\vec{X}$, if $\vec{Y}$ is

   (a) $a\vec{i} + 4\vec{j}$, (c) $4\vec{i} + a\vec{j}$,

   (b) $a\vec{i} - 4\vec{j}$, (d) $a\vec{i} - 3\vec{j}$.

4. Find the angle between $\vec{X}$ and $\vec{Y}$ in each part of Exercise 1 above.

5. If $a^2 + b^2 \neq 0$ prove that $a\vec{i} + b\vec{j}$ is perpendicular to $c\vec{i} + d\vec{j}$ if and only if $a\vec{i} + b\vec{j}$ is parallel to $-d\vec{i} + c\vec{j}$.

6. Find the component of $\vec{Y}$ in the direction of $\vec{X}$ if

   (a) $\vec{X} = \vec{i}$, $\vec{Y} = 3\vec{i} + 4\vec{j}$. (e) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = 3\vec{i} + 4\vec{j}$.

   (b) $\vec{X} = \vec{j}$, $\vec{Y} = 3\vec{i} + 4\vec{j}$. (f) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = 5\vec{i} + 2\vec{j}$.

   (c) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = \vec{i}$. (g) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = a\vec{i} + b\vec{j}$.

   (d) $\vec{X} = 3\vec{i} + 4\vec{j}$, $\vec{Y} = \vec{j}$. (h) $\vec{X} = p\vec{i} + q\vec{j}$, $\vec{Y} = a\vec{i} + b\vec{j}$.

11-5. Applications of Vectors in Physics.

The notion of "force" is one of the important concepts of physics. This is the abstraction which physicists have invented to describe "pushes" and "pulls" and to account for the effects that pushes and pulls produce.

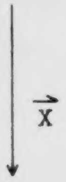
The student who knows something about vectors can readily learn about forces. The connecting links between the concepts of "force" and "vector" can be stated as follows:

1. Every force can be represented as a vector. The direction of the force is the same as the direction of its representative vector. The magnitude of the force determines the length of its representing vector, once a "scale" has been selected.

Example 11-5a: A red-headed man is standing on top of a hill, staring into space. He weighs 200 pounds. Represent as a vector each of the following:

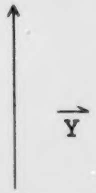
(a) the downward pull of the earth's gravity on him,
(b) the upward push of the hill on him.

Solution: (a)



Scale: 1 inch = 200 lbs.

(b)

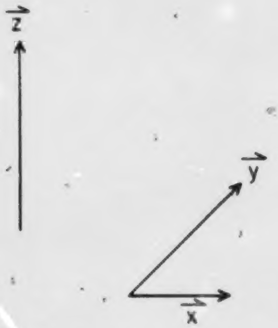


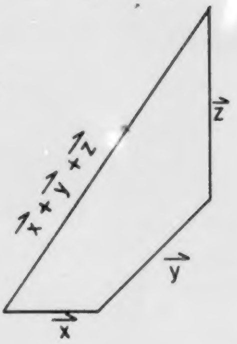
Scale: 1 inch = 200 lbs.

2. Any collection of forces which act on a single body is equivalent to a single force, called their resultant. If all the forces are represented as vectors on the same scale, then the vector which represents the resultant of the forces is the sum of these vectors.

Example 11-5b: Represent each of the following forces as a vector, and find the vector which represents their resultant: A force F_1 of 300 pounds directed to the right, a force F_2 of 400 pounds directed at an angle of 45° with the x-axis and a force of 500 pounds directed upward.

Solution: (graphical) Using the scale 1 inch = 400 pounds $\vec{X}$ represents F_1, $\vec{Y}$ represents F_2, $\vec{Z}$ represents F_3.





$\vec{X} + \vec{Y} + \vec{Z}$ represents the resultant of F_1, F_2, F_3. Its length is a little less than 5/2 inches; its direction is about 54°.

3. If F and G are two forces which have the same direction, then they have a numerical ratio and there is a number r such that r times force F is equivalent to force G. Moreover if $\vec{F}$ is the vector which represents force F, then $r\vec{F}$ is the vector which represents force G, where r is the ratio of force G to force F.

Example 11-5c: Emily and Elsie are identical twins. They are sitting on a fence. If $\vec{F}$ represents the total force Emily and Elsie exert on the fence and if $\vec{G}$ represents the force the fence exerts on Emily alone, express

(a) $\vec{F}$ in terms of $\vec{G}$.
(b) $\vec{G}$ in terms of $\vec{F}$.

Solution:

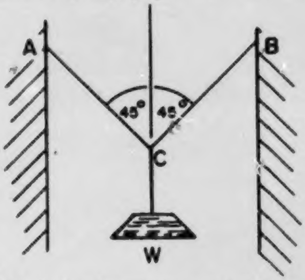
(a) $\vec{F} = -2\vec{G}$.

(b) $\vec{G} = -\frac{1}{2}\vec{F}$.

A body at rest is said to be in equilibrium. It is a fact of physics that if a body is at rest the resultant of all the forces acting on the body has magnitude zero. (Note: The converse of this is not true, since the resultant of all the forces which act on a moving body can also be zero. According to the laws of physics, if the sum of all the forces which act on a body is zero, then the body must be either at rest or it must move in a straight line with constant speed.)

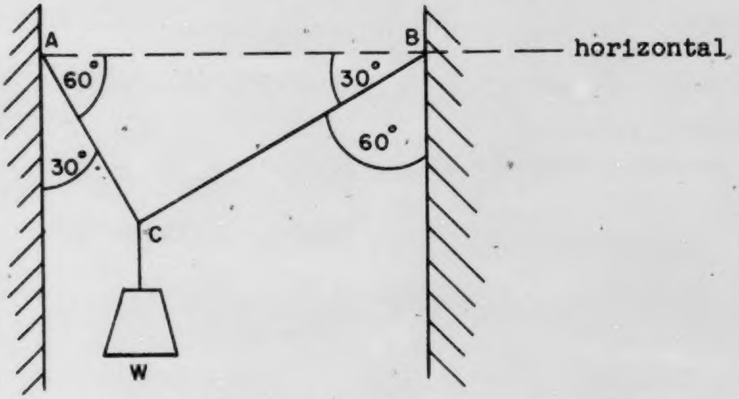
Exercises 11-5a

1. A weight is suspended by ropes as shown in the figure.



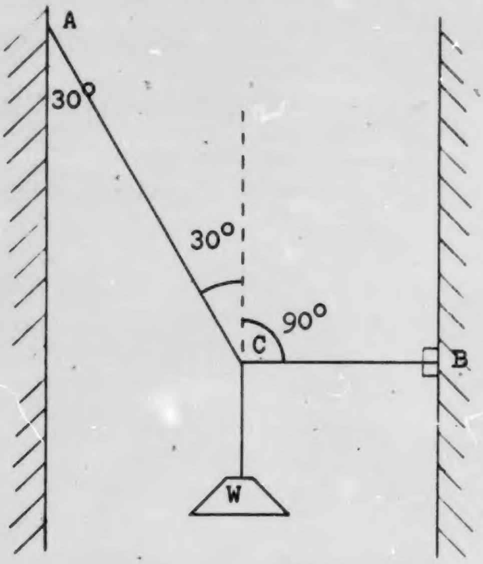
If the weight weighs 10 pounds, what is the force exerted on the junction C by the rope CB ?

2. A weight of 1,000 pounds is suspended from wires as shown in the figure.

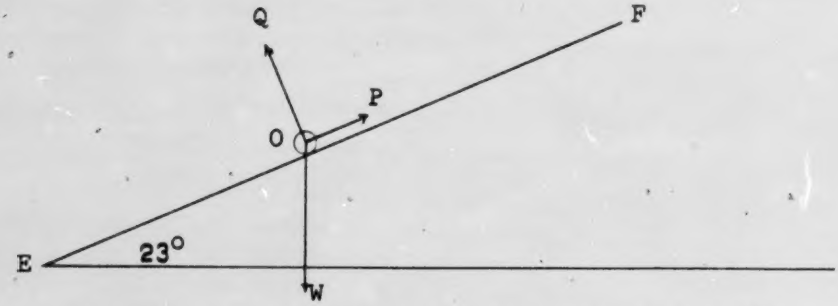


The distance $\overline{AB}$ is 20 feet, $\overline{AC}$ is 10 feet, and $\overline{CB}$ is $10\sqrt{3}$ feet. What force does the wire AC exert on the junction C ? What force does wire BC exert on C ? If all three wires are about equally strong, which wire is most likely to break? Which wire is least likely to break?

3. A 5000 pound weight is suspended as shown in the figure. Find the tension in each of the ropes CA, CB and CW.

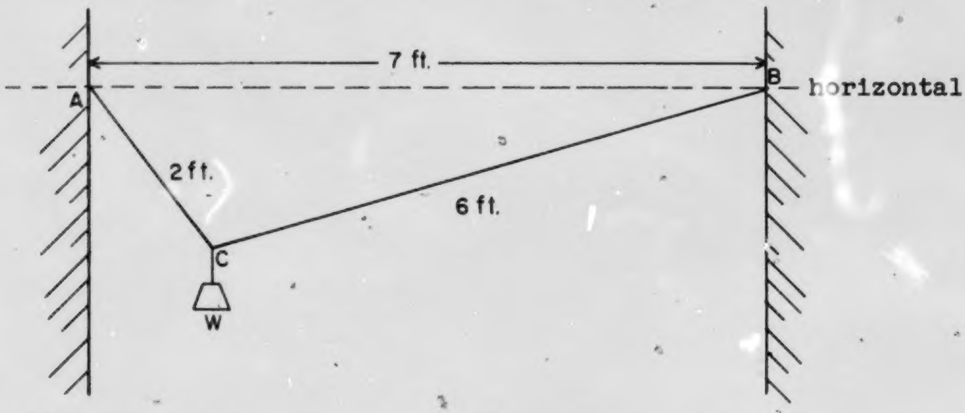


4. A barrel is held in place on an inclined plane EF by a force $\overrightarrow{OP}$ operating parallel to the plane and another operating perpendicular to it. (See diagram.)

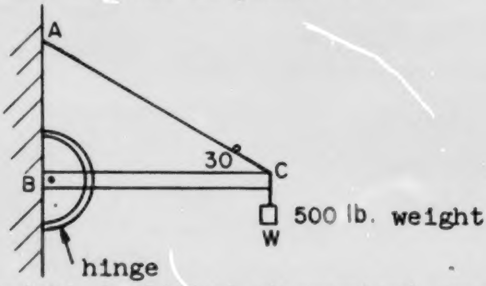


If the weight of the barrel is 300 pounds, ($|\overrightarrow{OW}| = 300$) and the plane makes an angle of 23° with the horizontal find $|\overrightarrow{OP}|$ and $|\overrightarrow{OQ}|$. (Hint: Introduce a coordinate system with origin at O and OW as negative y-axis.)

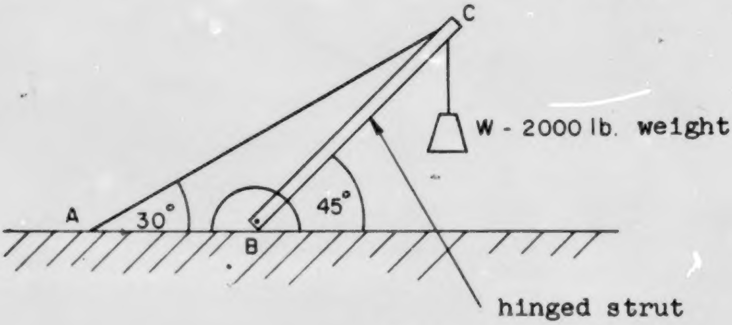
5. A weight is suspended by ropes as shown in the figure. If the weight weighs 20 pounds, what is the force exerted on the junction C by the rope CB? By the rope AC? If AC and CB are equally strong, which one is more likely to break?

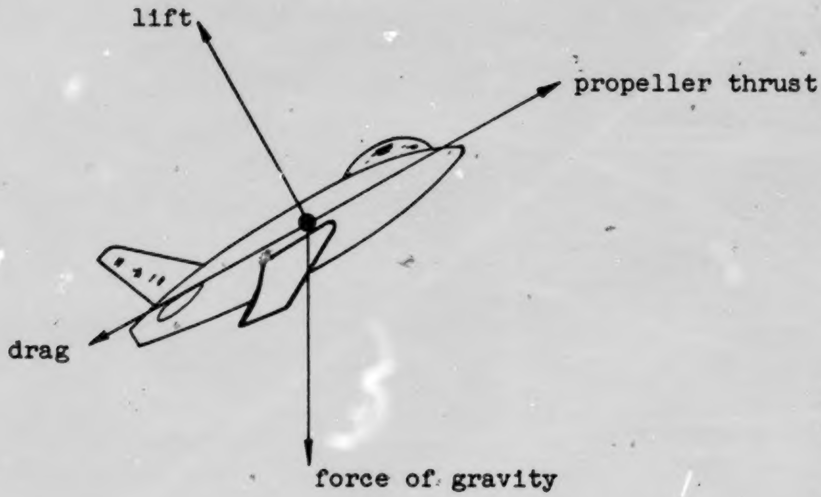

7 ft.
A
B
horizontal
2 ft.
6 ft.
C
W

6. A 500 pound weight is suspended as shown in the figure. Find each of the forces exerted on point C .


A
30°
C
B
500 lb. weight
W
hinge

7. A 2,000 pound weight is lifted at constant speed, as shown in the diagram. Find each of the forces exerted on point C .


C
W - 2000 lb. weight
A
30°
45°
B
hinged strut



The motion of airplanes provides another application of vectors. Some technical terms involved are listed and illustrated in the figure.

Lift: $\vec{F}_L$--a force perpendicular to the direction of motion. This is the "lifting force" of the wing.

Gravity: $\vec{F}_g$--a force directed downward.

Propeller thrust: $\vec{F}_{pt}$--a forward force in the direction of motion.

Drag: $\vec{F}_d$--a backward force parallel to the motion. This force is due to wind resistance.

Effective propeller thrust: $\vec{F}_{ept}$--the propeller thrust minus the drag.

The physical principle we shall use states that a body in motion will continue to move in a straight line with constant speed if and only if the resultant of all the forces acting on the body is zero.

8. An airplane weighing 6,000 pounds climbs steadily upwards at an angle of 30°. Find the effective propeller thrust and the lift.
9. An airplane weighing 10,000 pounds climbs at an angle of 15° with constant speed. Find the effective propeller thrust and the lift.
10. A motorless glider descends at an angle of 10° with constant speed. If the glider and occupant together weigh 500 pounds, find the drag and the lift.

The term "work" as the physicists use it also provides an example of a concept which can be discussed in terms of vectors. Consider for instance a tractor pulling a box-car along a track.

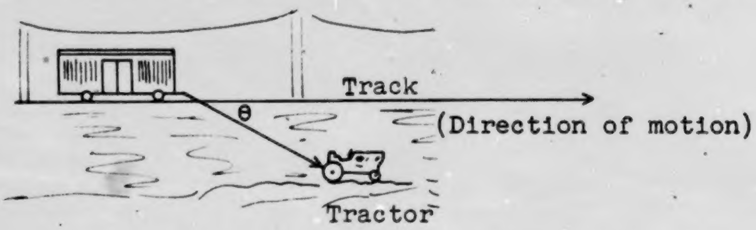


Fig. 11-5a

The effect of the tractor's force depends on the angle θ . It also involves the force itself and the displacement produced. The term "work," as used in physics, is given in this case by $\vec{F} \cdot \vec{S}$, where $\vec{F}$ is the force-vector of the tractor and where $\vec{S}$ is the displacement of the box car.

More generally, if a force F acts on a body and produces a displacement S while it acts, then the work done by the force is defined to be $\vec{F} \cdot \vec{S}$, where $\vec{F}$ is the vector which represents the force and where $\vec{S}$ is the vector which represents the displacement.

Example 11-5e: If the tractor of Figure 11-5a exerts a force of 1,000 pounds at an angle of 30° to the track, how much work does the tractor do in moving a string of cars 2,000 feet?

Solution: Evaluate the expression $|\overrightarrow{F}|\cdot|\overrightarrow{S}| \cos \theta$ where $|\overrightarrow{F}| = 1{,}000$ pounds, $|\overrightarrow{S}| = 2{,}000$ feet, $\cos \theta \approx .866$. The value of this product is approximately 1,732,000 foot pounds.

Exercises 11-5b

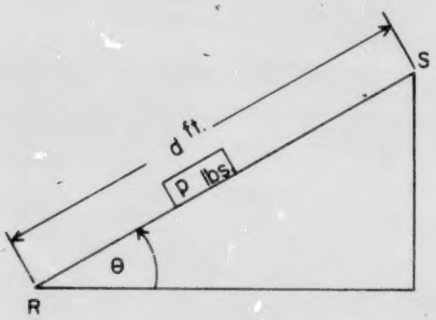
1. A sled is pulled a distance of d feet by a force of p pounds which makes an angle of θ with the horizontal. Find the work done if

   (a) $d = 10$ feet, $p = 10$ pounds, $\theta = 10^\circ$.
   (b) $d = 100$ feet, $p = 10$ pounds, $\theta = 20^\circ$.
   (c) $d = 1{,}000$ feet, $p = 10$ pounds, $\theta = 30^\circ$.

   How far can the sled be dragged if the number of available foot pounds of work is 1,000 and if

   (d) $p = 10$ pounds, $\theta = 10^\circ$.
   (e) $p = 100$ pounds, $\theta = 20^\circ$.
   (f) $p = 100$ pounds, $\theta = 0^\circ$.
   (g) $p = 100$ pounds, $\theta = 89^\circ$.

2. The drawing shows a smooth incline d feet long which makes an angle θ with the horizontal.



How much work is done in moving an object weighing p pounds from R to S if

(a) $d = 10$ feet, $p = 10$ pounds, $\theta = 10^\circ$.
(b) $d = 100$ feet, $p = 10$ pounds, $\theta = 20^\circ$.
(c) $d = 100$ feet, $p = 10$ pounds, $\theta = 30^\circ$.

How far can the weight be moved if the number of available foot pounds is 1,000 and if
(d) $p = 10$ pounds, $\theta = 10^\circ$.
(e) $p = 10$ pounds, $\theta = 20^\circ$.
(f) $p = 100$ pounds, $\theta = 1^\circ$.
(g) $p = 100$ pounds, $\theta = 89^\circ$.

Velocity is another concept of physics that can be represented by means of vectors. In ordinary language the words "speed" and "velocity" are used as synonyms. In physics the word "speed" refers to the actual time rate of change of distance (the kind of information supplied by an automobile speedometer), and "velocity" refers to the vector whose direction is the direction of the motion and whose length represents the speed on some given scale. When velocities are represented by vectors, the lengths of these vectors give the corresponding speeds.

Figure 11-5b shows vectors which represent some of the velocities of a body moving around a circle with constant speed.

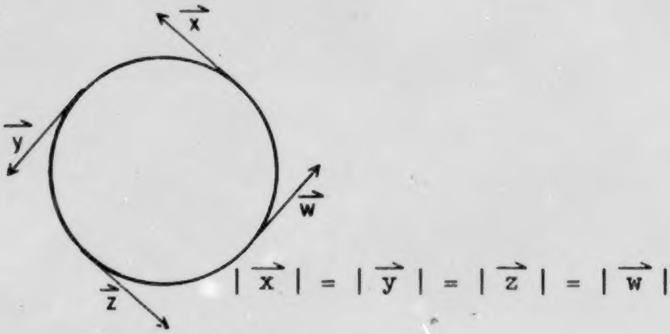
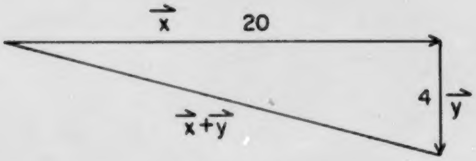


Fig. 11-5b

It is easy to imagine situations in which velocities are compounded out of other velocities. For instance, a man walking across the deck of a moving boat has a velocity relative to the water which is compounded out of his velocity relative to the boat and out of the boat's velocity relative to the water. It is a principle of physics that the vector which represents such a compound velocity is the sum of the vectors which represent the individual velocities.

Example 11-5f: A ship sails east at 20 miles per hour. A man walks across its deck toward the south at 4 miles per hour. What is the man's velocity relative to the water?

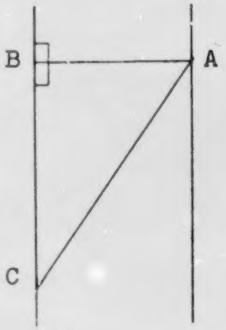


Solution: In the figure, $\vec{X}$ represents the ship's velocity relative to the water, $\vec{Y}$ represents the man's velocity relative to the ship. Consequently, $\vec{X} + \vec{Y}$ represents the man's velocity relative to the water. Its length is

$$\sqrt{20^2 + 4^2} \approx 20.4$$

and its direction is approximately 22° south of east.

Exercises 11-5c



1. A river 1 mile wide flows at the rate of 3 miles per hour. A man rows across the river, starting at A and aiming his boat toward B the nearest point on the opposite shore as shown in the diagram. If it took 30 minutes for him to make the trip, how far did he row?

2. A river is $\frac{1}{2}$ mile wide and flows at the rate of 4 miles per hour. A man rows across the river in 25 minutes, landing 1.3 miles farther downstream on the opposite shore. How far did he row? In what direction did he head?
3. A river one mile wide flows at a rate of 4 miles per hour. A man wishes to row in a straight line to a point on the opposite shore two miles upstream. How fast must he row in order to make the trip in one hour?
4. A body starts at (0,0) at the time $t = 0$. It moves with constant velocity, and 20 seconds later it is at the point (40,30). Find its speed and its velocity, if one unit of length of vector corresponds to 100 feet per second.
5. A body moves with constant velocity which is represented by the vector $\overrightarrow{V} = 10\overrightarrow{i} + 10\overrightarrow{j}$. If the body is at the point (0,1) at time $t = 2$, where will it be when $t = 15$? The scale is: One unit of length of vector corresponds to 10 miles per hour; the time t is measured in hours.
6. Ship A starts from point (2,4) at time $t = 0$. Its velocity is constant, and represented by the vector $\overrightarrow{V_a} = 4\overrightarrow{i} - 3\overrightarrow{j}$. Ship B starts at the point (-1,-1) at time $t = 1$. Its velocity is also constant, and is represented by the vector $\overrightarrow{V_b} = 7\overrightarrow{i} + \overrightarrow{j}$. Will the ships collide?

   (Assume that a consistent scale has been used in setting up the vector representation.)
7. Ship A starts at point (2,7) at time $t = 0$. Its (constant) velocity is represented by the vector $\overrightarrow{V_a} = 3\overrightarrow{i} - 2\overrightarrow{j}$. Ship B starts at point (-1,-1) at time $t = 1$. Its (constant) velocity is represented by the vector $\overrightarrow{V_b} = 5\overrightarrow{i}$. Will the ships collide?

*8. A river is 1/2 mile wide and flows at the rate of 4 miles per hour. A man can row at the rate of 3 miles per hour. If he starts from point A and rows to the opposite shore, what is the farthest point upstream at which he can reach the opposite shore? In what direction should he head?

Exercises 11-5d

1. Show each of the following graphically:

(a) $3\vec{i} + 8\vec{j} + 5\vec{k}$.
(b) $3\vec{j} + 3\vec{k}$.
(c) $4\vec{i} + 4\vec{j}$.
(d) $5\vec{i} + \vec{j}$.
(e) $5\vec{i} + 5\vec{j} + \vec{k}$.
(f) $2\vec{i} - 2\vec{j}$.
(g) $7\vec{k}$.
(h) $5\vec{j}$.
(i) $7\vec{i}$.
(j) $8\vec{i} + 8\vec{j} + 3\vec{k}$.

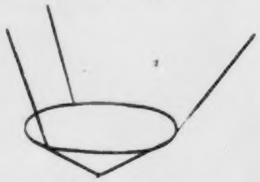
2. Find $\vec{A} \cdot \vec{B}$, if:

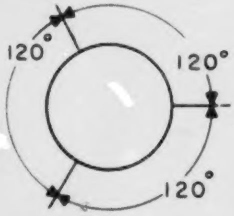
(a) $\vec{A} = 3\vec{i} + 2\vec{j} + 4\vec{k}$; $\vec{B} = 2\vec{i} + \vec{j} + 2\vec{k}$.
(b) $\vec{A} = 3\vec{i} + 4\vec{j} - 2\vec{k}$; $\vec{B} = 2\vec{i} + 2\vec{j} + 2\vec{k}$.
(c) $\vec{A} = 3\vec{i} + 3\vec{k}$; $\vec{B} = 4\vec{j}$.
(d) $\vec{A} = 4\vec{i} + 4\vec{j}$; $\vec{B} = 7\vec{k}$.
(e) $\vec{A} = 4\vec{j} + 2\vec{k}$; $\vec{B} = 5\vec{i}$.

3. Find the cosine of the angle between vectors $\vec{A}$ and $\vec{B}$ in each part of Problem 2.

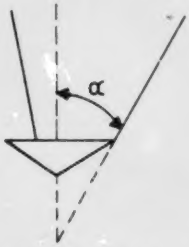
4. Find the cosine of the angle between the vectors $\vec{A}$ and $\vec{B}$ if $\vec{A} = 3\vec{i} + 2\vec{j} - \vec{k}$
and $\vec{B} = 4\vec{i} - 3\vec{j} + 6\vec{k}$.

*5. A lighting fixture is suspended as shown:





Side View
(the angle is
shown in its
true shape;
is an angle of
60°.)



The fixture weighs 15 pounds. Find the tension in each of the supporting cables.

6. An airplane is climbing at an angle of 30°. Its climbing speed is 100 m.p.h. Although a wind is blowing from west to east with a velocity of 30 m.p.h., the pilot wishes to climb while heading due north. What is the ground speed of the airplane?

7. Suppose that in Problem 6 the pilot climbs at an angle of 30°, but does not insist on heading north. What is the fastest ground speed that he can achieve? Which way should he head to achieve this speed? What is the least ground speed that he can achieve? Which way should he head to achieve this?

8. Prove that

$$a(x - d) + b(y - e) + c(z - f) = 0$$

is the equation of a plane through the point $Q(d,e,f)$ with the normal vector

$$\vec{N} = a\vec{i} + b\vec{j} + c\vec{k} .$$

9. Find a vector normal to the plane

$$7x - 3y + 5z = 12 .$$

10. Find the distance from the point $(0,0,0)$ to the plane

$$5x + 12y - z = 1 .$$

11. Find the distance from the plane

$$x + 2y - 3z = 1$$

to the origin.

11-6. Vectors as a Formal Mathematical System.

In our discussion of forces and velocities by means of vectors we made a few assumptions which we did not justify. We applied vector methods to the solution of force and velocity problems in a fashion which turns out to be correct but which we have not backed up with a convincing argument. Our thinking was something like this. "Some of the rules that forces obey are very much like the rules that vectors obey. Therefore we can talk about forces as though they were vectors." This is not really a sound argument, and if it were trusted in all cases it could lead to chaos. For instance, some of the rules that real numbers obey are the rules that integers obey, and it is not the case that real numbers can be regarded as integers.

Nevertheless, it really was correct to treat forces as vectors and we now explore a point of view which gives convincing evidence for this statement. The key fact in this examination is that every mathematical system which obeys certain of the laws which vectors obey must be essentially the same as the system of vectors itself.

We now formulate three goals:

1. To list the rules in question.
2. To give a precise specification of what we mean by saying that a mathematical system is "essentially the same" as a system of vectors.
3. To prove that systems which obey the stated rules are essentially the same as the system of vectors.

I. We state certain rules which vectors have been shown to obey. We have a set S, two operations $\oplus$, $\odot$, for which, for all α, β, γ, in S and for all real numbers r, s

(1) $\alpha \oplus \beta$ is in S.

(2) $\alpha \oplus \beta = \beta \oplus \alpha$.

(3) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.

(4) There is a zero element ϕ in S such that

$$\alpha \oplus \phi = \alpha .$$

(5) Each α has an additive inverse $-\alpha$ for which

$$\alpha \oplus (-\alpha) = \phi .$$

(6) $r \odot \alpha$ is in S .
(7) $r \odot (s \odot \alpha) = (rs) \odot \alpha$.
(8) $(r + s) \odot \alpha = (r \odot \alpha) \oplus (s \odot \alpha)$.
(9) $r \odot (\alpha \oplus \beta) = (r \odot \alpha) \oplus (r \odot \beta)$.
(10) $1 \odot \alpha = \alpha$.
(11) There are two members y and w of S such that each α has a unique representation

$$\alpha = (a \odot y) \oplus (b \odot w) .$$

II. We have already shown that vectors satisfy such rules, where S is interpreted as the set of vectors, $\oplus$ is interpreted as ordinary $+$ for vectors and where $\odot$ is interpreted as scalar multiplication. We take it as given (by physicists presumably) that forces also satisfy these rules, where S is the set of forces, $\alpha \oplus \beta$ means the resultant of α and β and $\odot$ means scalar multiplication. We are to show that forces are essentially the same as vectors. What do we mean by "essentially the same?" We mean that the system of forces is isomorphic to the system of vectors. What do we mean by "isomorphic"? That there is a one-to-one correspondence between the set of forces and the set of vectors such that, if force α corresponds to vector $\overrightarrow{A}$ and if force β corresponds to vector $\overrightarrow{B}$, then $\alpha \oplus \beta$ corresponds to vector $\overrightarrow{A} + \overrightarrow{B}$ and force $r \odot \alpha$ corresponds to vector $r\overrightarrow{A}$.

III. We now state and prove the promised theorem.

Theorem: Any system S which satisfies Rules 1-11 is isomorphic to the system of vectors in a plane.

Proof: We first set up a one-to-one correspondence between the members of S and the vectors. For each α of S we invoke Item 11 to write

$$\alpha = (a \odot y) \oplus (b \odot w) .$$

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The pair (a,b) which figures in this expression determines a unique vector $\vec{A}$, namely $[a,b]$, which we pair with α. This process assigns to each α of S a vector $\vec{A}$ as its image. We must show that if $[a,b]$ is the image of α and if $[c,d]$ is the image of β, then $[a + c,\ b + d]$ is the image of $\alpha + \beta$ and that $[ra,rb]$ is the image of $r \odot \alpha$. To prove the first, write

$$\alpha = (a \odot y) \oplus (b \odot w)$$

$$\beta = (c \odot y) \oplus (d \odot w)\ .$$

Therefore $\alpha \oplus \beta = ((a \odot y) \oplus (b \odot w)) \oplus ((c \odot y) \oplus (d \odot w))$ which equals using Rules 2 and 3,

$$((\alpha \odot y) \oplus (c \odot y)) \oplus ((b \odot w) \oplus (d \odot w))\ .$$

This in turn equals

$$((\alpha + c) \odot y) \oplus ((b + d) \odot w)$$

by virtue of Rule 8. We see then that our one-to-one correspondence assigns $[a + c,\ b + d]$ to $\alpha + \beta$.

We now examine $r \odot \alpha$. We write

$$r \odot \alpha = r \odot ((a \odot y) \oplus (b \odot w))$$

which by Rule 9 can be written as

$$r \odot (a \odot y) \oplus r \odot (b \odot w)\ .$$

According to Rule 7, this last equals

$$((ra) \odot y) \oplus ((rb) \odot w)\ ,$$

whence the image of $r \odot \alpha$ is indeed $[ra,rb]$.

This completes our proof. Notice that we did not use all the rules given. They are in fact redundant. If the last rule is left out, the remaining set of rules is not redundant, and is the set of axioms which defines a vector space. The Rules 1-11 are axioms for a more special mathematical system--a two-dimensional vector space.

We have shown that every system which satisfies Rules 1-11 is isomorphic to our system of vectors. We have not shown that the system of forces satisfies these rules. We take the physicist's word for this. We have not shown that to be "isomorphic" really means to be "essentially the same." Let us meditate a little on this and then take the mathematician's word for it.

Exercises 11-6

1. Let S be the system of complex numbers. Does S satisfy Rules 1-11 if $\oplus$ is interpreted as ordinary addition of complex numbers and $\odot$ as ordinary multiplication of a real number by a complex number. (Hint: In checking Rule 11 try 1 for y and i for w).
2. Let S be the set of all ordered pairs (a,b) of real numbers, let $\oplus$ be defined by $(a,b) \oplus (c,d) = (a + c, b + d)$ and let $\odot$ be defined by

$$r \odot (a,b) = \left(\frac{ra}{2}, \frac{rb}{2}\right) .$$

   Which of the Rules 1-11 does this system obey?

3. Let S be the set of all ordered pairs (a,b) of real numbers, let $\oplus$ be defined by $(a,b) \oplus (c,d) = \left(\frac{a + c}{2}, \frac{b + d}{2}\right)$, and let $\odot$ be defined by $r \odot (a,b) = (ra,rb)$. Which of the Rules 1-11 does this system obey?

Chapter 12

POLAR FORM OF COMPLEX NUMBERS

12-1. Introduction.

In Chapter 5 we introduced complex numbers $z = x + iy$, x and y real numbers. We found (Theorem 5-4) that each complex number z is uniquely determined by its "real" and "imaginary" parts, x and y, respectively; i.e.,

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2 \text{ are equal}$$

$$\text{if and only if } x_1 = x_2 \text{ and } y_1 = y_2 .$$

We also discussed the addition and multiplication of complex numbers given by the formulas:

12-1a $$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) ,$$

12-1b $$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) .$$

We found in Section 5-7 that the addition of complex numbers may be described geometrically by means of a parallelogram. In Section 12-2 we discuss a geometrical description of the product of two complex numbers.

The remainder of this section points out some similarities between the work in Chapters 5 and 11. Exercises 12-1 provide a review of some of the work in Chapter 5.

Complex Numbers and Vectors. We call attention to the important case of Formula 12-1b in which $y_1 = 0$:

12-1c $$x_1(x_2 + iy_2) = (x_1x_2) + i(x_1y_2) \ .$$

In view of Chapter 11, this special case appears in a new light. Note the similarity between Theorem 5-4 and Formulas 12-1a, 12-1c and the statements in Chapter 11 concerning equality, sum, and scalar multiple of vectors in a plane.

Just as two complex numbers are equal if and only if their real and imaginary parts are the same, two vectors in a plane are equal if and only if their x and y components are the same. This similarity is more than a coincidence: our geometrical representation of complex numbers is exactly the same as our pictures of vectors in a plane.

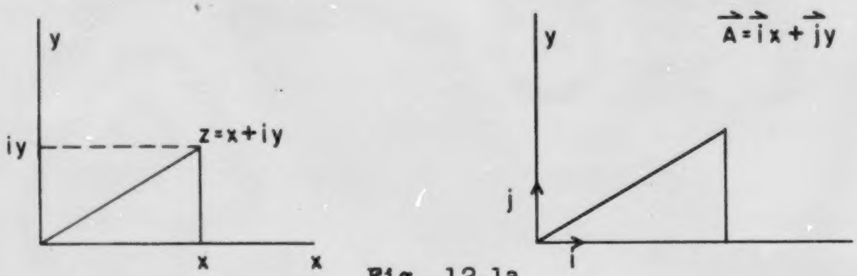


Fig. 12-1a

Moreover we add complex numbers just as we add vectors and we use the same picture (a parallelogram) to represent sums in each case.

Multiplication of complex numbers by real numbers, as in Formula 12-1c, corresponds exactly to the multiplication of vectors in a plane by scalars: we multiply each "component" by the real multiplier.

We thus recognize a kind of identity between these topics. It is true that we have used a different set of words in what we have said on these two subjects, but our formulas show that even with this difference in the words we have actually been saying precisely the same things in two different contexts.

Two mathematical systems which are the same in this sense are often called abstractly identical or isomorphic. (The word "isomorphic" has the Greek roots "iso," meaning "same," and "morphos," meaning "shape" or "form." See page 680.)

It must be emphasized that our isomorphism is between fragments of these two subjects. The theory of complex numbers and the theory of vectors in a plane have the same form only when we restrict our attention to the notion of equality and the operations of addition and multiplication by a real number (scalar), and to ideas depending solely on these.

Isomorphism--like analogy--is not necessarily complete identity. Our two systems--vectors in a plane and complex numbers--differ remarkably, and in two very important respects. They differ when it becomes a matter of discussing an operation of "multiplication" between elements of the two systems: product of two complex numbers, and product of two vectors. Perhaps the most startling difference between the products in our two systems is the matter of closure. The product of two complex numbers is a complex number, the product of two vectors in a plane is not a vector in the plane. In the case of the inner product, it is not a vector of any kind, it is a scalar. Multiplication of complex numbers is associative. The question of associativity for the inner product of vectors is ludicrous; the very expressions

$$\vec{A} \cdot (\vec{B} \cdot \vec{C}) \text{ , } (\vec{A} \cdot \vec{B}) \cdot \vec{C}$$

whose equality is presumably at issue do not make sense since the factors in parentheses, being scalars, cannot be "dotted" onto a vector. Only a vector can be "dotted" onto another vector.

In Chapter 13, we discussed the geometric interpretation of the inner product of vectors. In Section 12-2 we pursue the analogous question for the product of complex numbers. Such an interpretation will serve to emphasize the differences we have been discussing.

The two systems also differ fundamentally if one attempts to extend them from a plane to a space of three or more dimensions. Vectors in spaces having $3, 4, \ldots, 6N_0, \ldots$ dimensions have very important applications in physics, chemistry and engineering. (N_0 is Avogadro's number, 6.025×10^{23}.) On the other hand, extensions of the system of complex numbers are a bit bizarre and, in any case, are another matter entirely. They are beyond the scope of this book.

Exercises 12-1. (Review of Chapter 5.)

1. Write the following in standard form:
   (a) 1 (d) $1 + i^2$
   (b) i (e) $\sqrt{-16}$
   (c) $\frac{2}{13} - \frac{3}{13}i$ (f) $\sqrt{-7}$
2. Write the conjugate for each of the following complex numbers:
   (a) $2 - 3i$
   (b) $5 + i$
   (c) $-2 + 3i$
3. If $z = a + bi$, express $z \cdot \bar{z}$ in standard form.
4. Express the quotient $\frac{2 - 3i}{3 + 2i}$ in standard form.

5. Find the absolute value of:

   (a) $4 + 3i$

   (b) $-2 - 5i$

   (c) $-3i$

6. Solve the equation $2z^2 + z + 1 = 0$.

7. Plot each of the following complex numbers in an Argand Diagram:

   (a) $2 + i$

   (b) $-3 + 2i$

   (c) $3i$

   (d) 5

   (e) $-5 - 3i$

8. Find the sum, difference, or product as indicated:

   (a) $(2 - 3i) + (-4 + i)$

   (b) $(-3 + 2i) + (1 - i)$

   (c) $(2 + 3i) - (4 - 2i)$

   (d) $5 - (4 - 2i)$

   (e) $5 + (2 - i)$

   (f) $3(2 - 3i)$

   (g) $(2 + 3i)(1 - i)$

   (h) $(1 + 2i)(1 - 2i)$

9. If a and b are real numbers, under what conditions will $a + bi = b + ai$?

10. Solve each of the following for the real numbers x and y :

    (a) $(x + iy) + (2 - 3i) = 4 - i$

    (b) $2(x + iy) - (3 - 2i) = 1$

    (c) $\dfrac{x + iy}{3 - 5i} = -1 + 2i$

    (d) $(2x + i)(8 - ix) = 34$

11. Find the value of $\dfrac{1}{a + bi}$ by the following alternate methods:

    (a) $\dfrac{1}{a + bi}$ is a number $x + yi$ such that $(x + yi)(a + bi) = 1$. From this, obtain two equations for x , y , and solve them.

(b) $\frac{1}{a + bi}$ may be expressed as a number $x + yi$ in standard form by making use of the conjugate of $a + bi$.

12. Simplify: $\frac{1}{2i}\left(\frac{1}{z - i} - \frac{1}{z + i}\right)$.

12-2. Products and Polar Form.

In this section we consider the problem of a geometrical representation for the product of two complex numbers. We shall find some of the ideas and methods in Chapter 10 very useful in solving this problem. Moreover the introduction of trigonometrical notions enables us to write complex numbers in a form particularly convenient for the study of powers and roots given in the remaining sections of this chapter.

We know from Chapter 5 that the absolute value of a product of two complex numbers is the product of their absolute values:

$$|z_1 z_2| = |z_1| \cdot |z_2| .$$

In view of Formula 12-1b, this follows easily from the identity

$$(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 = (x_1{}^2 + y_1{}^2)(x_2{}^2 + y_2{}^2) .$$

This fact alone tells us something rather interesting about the product. Suppose z_1 represents a point on the circle with center 0 and radius r_1. Then $r_1 = |z_1|$. If z_2 represents a point on the circle with center 0 and radius r_2, then $r_2 = |z_2|$, and the product $z_1 z_2$ represents a point on the circle with center 0 and radius $r_1 r_2$, since $r_1 r_2 = |z_1 z_2|$.

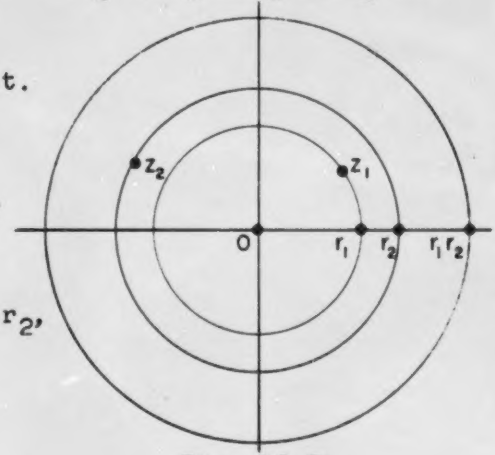


Fig. 12-2a

Given that z_1z_2 represents some point on the circle, our problem is now to locate which point on the circle.

We discussed questions of this kind in Chapter 10. In Figure 12-2b we reproduce the fundamental diagram from Chapter 10, drawn in an Argand diagram. As in Chapter 10 we have

$$x = r \cos \theta \ ,$$

$$y = r \sin \theta \ ,$$

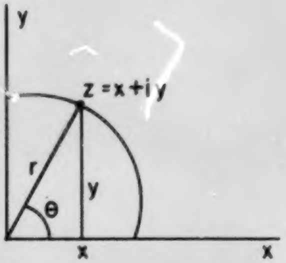


Fig. 12-2b

so, if $z = x + iy$ is in standard form we can write

12-2a $$z = r(\cos \theta + i \sin \theta) \ ,$$

where $r = |z| = \sqrt{x^2 + y^2}$.

Formula 12-2a, expressing z in terms of r and θ enables us to solve our problem. Indeed, suppose

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \ ,$$

$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \ .$$

If we form the product of these expressions, we obtain

$$z_1z_2 = r_1r_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)$$

12-2b $$= r_1r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \ .$$

Using the addition theorems for cosine and sine (Chapter 10),

$$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$$

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \ ,$$

we can simplify 12-2b.

Thus

12-2c $$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] .$$

Formula 12-2c gives the following geometrical description of the product of a pair of complex numbers: to multiply two complex numbers one multiplies their absolute values and adds their angles.

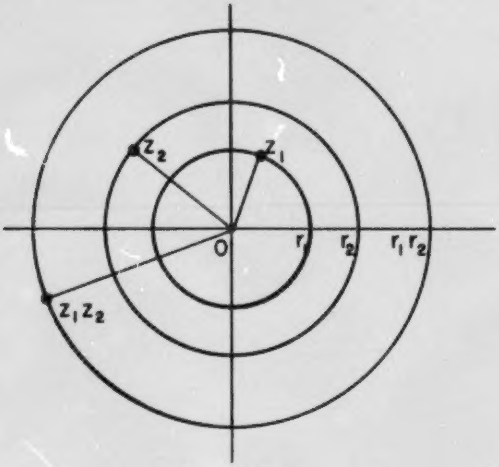


Fig. 12-2c

Formula 12-2a, expressing the complex number z in terms of r and θ, is called the polar form of z. We have seen that Formula 12-2c gives us a way to describe a product in geometrical terms. We shall also see that the algebraic consequences of these formulas are extremely important.

Example 12-2a: Multiply $3i$ and $1 + i$, plot the product and the factors, and check the result using polar forms of the numbers involved.

Solution: $3i(1 + i) = 3i - 3 = -3 + 3i$

$$|3i| = 3\ ,\ |1 + i| = \sqrt{2}\ ,\ |-3 + 3i| = 3\sqrt{2}\ .$$

Polar forms: $3i = 3(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$

$$1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

$$-3 + 3i = 3\sqrt{2} \cos(\frac{\pi}{2} + \frac{\pi}{4}) + i \sin(\frac{\pi}{2} + \frac{\pi}{4})$$

$$= 3\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})\ .$$

Example 12-2b: $(1 + i)(1 - i) = 2$.

$$1 + i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})\ ,$$

$$1 - i = \sqrt{2}[\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})]\ ,$$

$$(1 + i)(1 - i) = (\sqrt{2})^2[\cos(\frac{\pi}{4} - \frac{\pi}{4}) + i \sin(\frac{\pi}{4} - \frac{\pi}{4})]$$

$$= 2(\cos 0 + i \sin 0)$$

$$= 2\ .$$

Let us examine the relation between the standard form of z and the polar form of z , $(z \neq 0)$. For the standard form we write

$$z = x + iy \qquad (x \text{ and } y \text{ real})\ ;$$

for the polar form we write

$$z = |z|(\cos \theta + i \sin \theta)\ .$$

Since

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{|z|}\ ,\ \sin \theta = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{|z|}\ ,$$

the polar form, expressed in terms of x and y , is simply

$$z = |z|(\frac{x}{|z|} + i \frac{y}{|z|})\ .$$

The polar form of z may be described by saying that it resolves z into a product of two factors: the first factor being $|z|$, a non-negative real number, the second factor $\cos\theta + i\sin\theta = \frac{x}{|z|} + i\frac{y}{|z|}$, which is complex, has absolute value 1:

$$|\cos\theta + i\sin\theta| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1 .$$

__Example__ __12-2c__: The polar form of $1 + i$ is

$$\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

since $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$, $\frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$.

__Example__ __12-2d__: The polar form of $3 + 4i$ is

$$5\left(\frac{3}{5} + i\frac{4}{5}\right) = 5(\cos\theta + i\sin\theta),$$

where $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$

These examples illustrate the fact that one does not have to refer to a table of trigonometric functions in order to write the polar form of a complex number when it is given in standard form. Reference to a table is necessary only for determining, or estimating, the value of θ. We shall see that for many calculations involving the polar form it is not necessary to find θ itself --knowing only $\cos\theta$ and $\sin\theta$ being sufficient.

It is clear, on geometrical grounds, that there are many values of θ corresponding to each given non-zero complex number z However, these values of θ are related to each other in a very simple way since each of them measures an angle from the positive x-axis to the ray from 0 through the point representing z. Such angles differ by some number of complete revolutions about 0, so the values of θ measuring these angles differ by integral multiples of 2π. Each ray from 0 is terminal side of one angle, having the positive x-axis as initial side, which is less than one complete revolution. If θ is the radian measure of such an angle then $0 \leq \theta < 2\pi$; we shall say that θ is the *argument* of each non-zero complex number z corresponding to a point on the ray. We write $\theta = \arg z$. Thus $\theta = \arg z$ means

$$z = r(\cos\theta + i\sin\theta), \; r > 0, \text{ and } 0 \leq \theta < 2\pi .$$

Since $|\cos\theta + i\sin\theta| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1$, the complex number $\cos\theta + i\sin\theta$ represents a point on the "unit circle"--i.e., the circle with center 0 and radius 1. Consider two such complex numbers: $\cos\phi + i\sin\phi$ and $\cos\theta + i\sin\theta$. By the remarks in the previous paragraph, we have the following theorem.

Theorem 12-2a: $\cos\phi + i\sin\phi = \cos\theta + i\sin\theta$

if and only if

$\phi = \theta + 2k\pi$, for some integer k.

This theorem may be proved directly from the periodicity properties of the cosine and sine functions without appeal to geometrical ideas. See Exercise 12-2.

Exercises 12-2

1. Express each of the following complex numbers in polar form and determine $\arg z$:

(a) $z = 2 + 2i$ (d) $z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(b) $z = -3 + 3i$ (e) $z = 4$

(c) $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ (f) $z = -2i$

2. Express each of the following complex numbers in standard form $a + bi$:

(a) $3(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$ (d) $5(\cos \pi + i \sin \pi)$

(b) $2(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$ (e) $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

(c) $\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}$ (f) $2(\cos 0 + i \sin 0)$

3. Find the indicated products in polar form and express them in standard form:

(a) $2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \cdot 3(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})$

(b) $(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}) \cdot (\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$

(c) $[3(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})]^2$

4. Prove that $[r(\cos \theta + i \sin \theta)]^2 = r^2(\cos 2\theta + i \sin 2\theta)$, where r is a real number.

5. Prove that, if $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, r_1 real,

and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \neq 0$, r_2 real,

then $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$.

6. Show that $\bar{z} = \frac{1}{z}$ if $|z| = 1$.

7. Show that two non-zero complex numbers lie on the same ray from 0 if and only if their ratio is a positive real number.
8. Prove Theorem 12-2a without appealing to geometry; i.e., show that

   (a) $\cos \phi + i \sin \phi = \cos \theta + i \sin \theta$

   if and only if $\phi = \theta + 2k\pi$, where k is some integer.

   [Hint: Prove first that (a) holds if and only if

   (b) $$\frac{\cos \phi + i \sin \phi}{\cos \theta + i \sin \theta} = 1 .$$

   Rewrite (b) using Exercise 5; equate real and imaginary parts and show that the two conditions you get hold if and only if $\cos(\phi - \theta) = 1$. But the last condition holds if and only if $\phi - \theta = 2k\pi$, for some integer k. Why?]

12-3. Integral Powers; Theorem of deMoivre.

We saw in Section 12-2 that if

$$z = r(\cos \theta + i \sin \theta)$$

and
$$z' = r'(\cos \theta' + i \sin \theta'),$$

then
$$zz' = rr'[\cos(\theta + \theta') + i \sin(\theta + \theta')] .$$

We now turn to the case where z and z' are equal and obtain for the square of a complex number

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta) .$$

We can extend the idea to

$$\begin{aligned} z^3 &= z^2 z \\ &= [r^2(\cos 2\theta + i \sin 2\theta)][r(\cos \theta + i \sin \theta)] \\ &= r^3(\cos 3\theta + i \sin 3\theta) . \end{aligned}$$

Continuing in this way, we may derive, one after the other, similar formulas for z^4, z^5, z^6, ... , z^n , for each natural number n . The theorem of de Moivre states the general result.

Theorem 12-3a: (de Moivre) If $z = r(\cos\theta + i\sin\theta)$ and n is a natural number, then

$$z^n = r^n(\cos n\theta + i\sin n\theta) .$$

Let us turn now to some special instances of this theorem to see what it has to tell us about the geometry of the complex plane.

Example 12-3a: Find all positive integral powers of i .

Solution: We have $|i| = 1$ and $\arg i = \frac{\pi}{2}$. Thus

$$i^2 = 1(\cos\pi + i\sin\pi) = -1 + 0i = -1 .$$

$$i^3 = 1(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 0 - i = -i .$$

$$i^4 = 1(\cos 2\pi + i\sin 2\pi) = 1 + 0i = 1 .$$

From here on the powers repeat: $i^5 = i^4i = i$, $i^6 = i^4i^2 = -1$, $i^7 = i^4i^3 = -i$, $i^8 = i^4i^4 = 1$, etc. We can explain this repetition in geometrical terms by noting that each time the exponent is increased by 1 , i^n steps through a quadrant of the unit circle. We can express these facts compactly by writing

$$i^{4n} = 1 , \quad i^{4n+1} = i , \quad i^{4n+2} = -1 , \quad i^{4n+3} = -i ,$$

where n is 0 or any natural number.

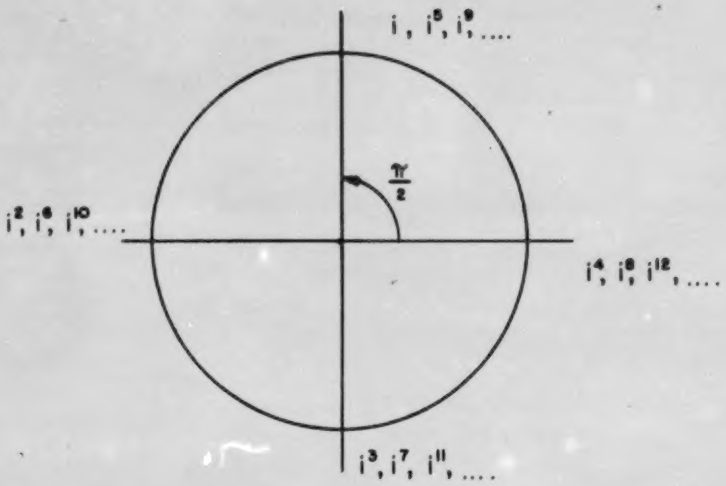

$i, i^5, i^9, \ldots$
$\frac{\pi}{2}$
$i^2, i^6, i^{10}, \ldots$
$i^4, i^8, i^{12}, \ldots$
$i^3, i^7, i^{11}, \ldots$

Fig. 12-3a

Example 12-3b: Let $z = \cos 1 + i \sin 1$. (The angle with radian measure 1 has degree measure $\frac{180}{\pi}$, which is approximately 57.3 degrees.) Plot the first ten powers of z . Which quadrant contains the point represented by z^{100} ?

Solution: Since

$$z = \cos 1 + i \sin 1 \text{ , de Moivre's Theorem gives}$$

$$z^n = \cos n + i \sin n \text{ , } n = 1, 2, 3, \ldots \text{ .}$$

These numbers all have absolute value 1 and hence represent points on the unit circle. The length of arc along the circle between successive powers of z is 1 unit. To determine the quadrant containing z^{100} we may first determine how many complete

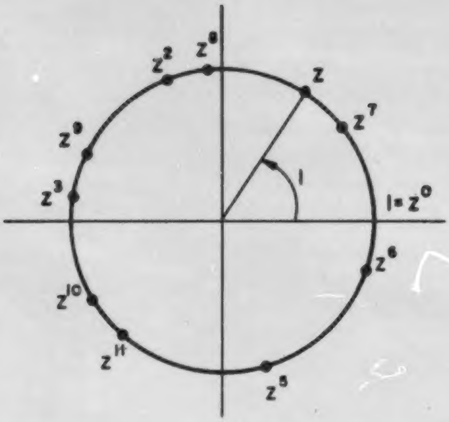


Fig. 12-3b

circuits occur as n steps from 0 to 100 . Dividing 100 by 2π we obtain 15.915... . Thus z^n steps through 15 complete revolutions and over $\frac{9}{10}$ of another as n steps from 0 to 100. Therefore, z^{100} represents a point in the fourth quadrant.

Example 12-3c: Calculate and plot the first five powers of $z = 1 + i\sqrt{3}$.

Solution: Here $|z| = \sqrt{1 + 3} = 2$, $\arg z = \frac{\pi}{3}$. So

$$z = 2(\cos \tfrac{\pi}{3} + i \sin \tfrac{\pi}{3}) = 1 + i\sqrt{3} ,$$

and
$$z^2 = 4(\cos \tfrac{2\pi}{3} + i \sin \tfrac{2\pi}{3}) = 4(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}) = -2 + 2i\sqrt{3} .$$

$$z^3 = 8(\cos \pi + i \sin \pi) = 8(-1 + 0 \cdot i) = -8 .$$

$$z^4 = 16(\cos \tfrac{4\pi}{3} + i \sin \tfrac{4\pi}{3}) = 16(-\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}) = -8 - 8i\sqrt{3}$$

$$z^5 = 32(\cos \tfrac{5\pi}{3} + i \sin \tfrac{5\pi}{3}) = 32(\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}) = 16 - 16i\sqrt{3}$$

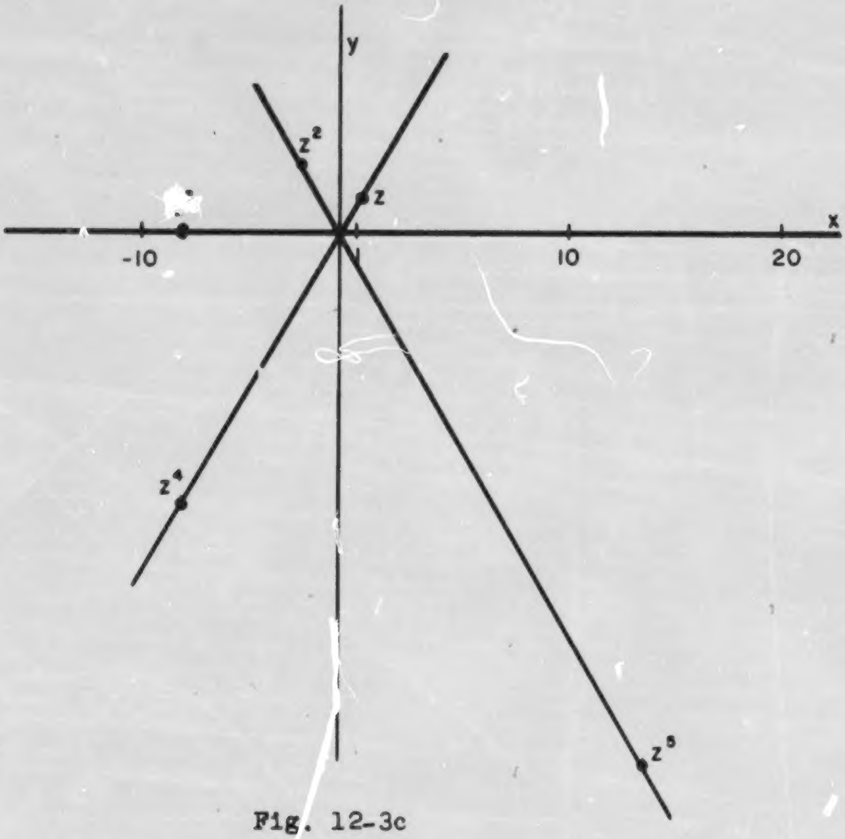


Fig. 12-3c

Using the theorem of de Moivre (Theorem 12-3a) and the formulas

$$(a + b)^2 = a^2 + 2ab + b^2 ,$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 ,$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 ,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

we may derive an endless list of identities of the following kind:

12-3a $\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2$,

12-3b $\sin 2\theta = 2(\cos\theta)\sin\theta$,

12-3c $\cos 3\theta = (\cos\theta)^3 - 3\cos\theta(\sin\theta)^2$,

12-3d $\sin 3\theta = 3(\cos\theta)^2\sin\theta - (\sin\theta)^3$,

12-3e $\cos 4\theta = 8(\cos\theta)^4 - 8(\cos\theta)^2 + 1$,

12-3f $\sin 4\theta = 8(\cos\theta)^3\sin\theta - 4(\cos\theta)\sin\theta$,

and corresponding formulas for $\cos 5\theta$, $\sin 5\theta$, $\cos 6\theta$, $\sin 6\theta$, ... , $\cos n\theta$, $\sin n\theta$, where n is any natural number. We call such identities multiplication formulas.

We may prove the identities 12-3c and 12-3d as follows:

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3\\ &= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta)\\ &\quad + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3\\ &= [(\cos\theta)^3 - 3(\cos\theta)(\sin\theta)^2]\\ &= i[3(\cos\theta)^2\sin\theta - (\sin\theta)^3] ,\end{aligned}$$

and, equating real and imaginary parts,

$$\begin{aligned}\cos 3\theta &= (\cos\theta)^3 - 3\cos\theta(\sin\theta)^2\\ \sin 3\theta &= 3(\cos\theta)^2\sin\theta - (\sin\theta)^3 .\end{aligned}$$

We leave the proofs of the other multiplication formulas as an exercise.

Exercises 12-3

In each of the exercises 1 through 5,

(a) Find $|z|$, $\arg z$, and express z in polar form.

(b) Using the polar form found in Step (a), calculate z^2 , z^3 , z^4 .

(c) Check the results obtained in Step (b) by calculating z^2, z^3, z^4 using the standard form.

(d) Show in a diagram each of the points 1, z, z^2, z^3, z^4.

1. $z = 1 + i$.
2. $z = \frac{-1 + i\sqrt{3}}{2}$.
3. $z = \frac{1}{3}$.
4. $z = 3 + 4i$.
5. $z = \frac{-1 - i\sqrt{3}}{2}$.
6. Deduce de Moivre's theorem for negative integral exponents from the version stated in the text for exponents which are natural numbers.
7. Prove the multiplication formulas 12-3a, 12-3b, 12-3e, 12-3f.

12-4. Square Roots.

The theorem of de Moivre in Section 12-3 provides a compact formula for any integral power of a non-zero complex number:

> If $z = r(\cos\theta + i\sin\theta)$, $r > 0$, and n is any integer, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

In Sections 12-4 and 12-6, we consider the converse problem:

> Given a complex number z and a natural number n, to find all complex numbers w satisfying the equation $w^n = z$.

Section 12-4 is devoted to the case $n = 2$. Section 12-6 contains the general theorems.

We recall that every non-zero real number has two square roots. If x is a positive real number, its square roots are real; we have denoted them by $\sqrt{x}$ and $-\sqrt{x}$ ($\sqrt{x}$ being the one which is positive, $-\sqrt{x}$ the negative one). In Chapter 5, we extended this notation to cover square roots of negative real numbers. Thus for $x < 0$,

$$\sqrt{x} = i\sqrt{|x|}, \qquad -\sqrt{x} = -i\sqrt{|x|} .$$

Let us consider the equation

$$w^2 = z ,$$

where z is a given non-zero complex number (which we may suppose is not real). We are interested in the solution set of this equation. Let us assume that the solution set is not empty and write

$$w = |w| (\cos \phi + i \sin \phi)$$

for one of its elements. If $z = |z| (\cos \theta + i \sin \theta)$, we have, by de Moivre's theorem and the assumption $w^2 = z$,

$$|w|^2 (\cos 2\phi + i \sin 2\phi) = |z| (\cos \theta + i \sin \theta) .$$

Equating absolute values we have

$$|w|^2 = |z| ;$$

so that

$$|w| = \sqrt{|z|} .$$

Note that $|w|$ is uniquely determined: $|z|$ is a positive real number and $|w|$, being a positive real number, is its positive square root. Knowing $|w|$, we must still find ϕ in order to get w. We have

$$\cos 2\phi + i \sin 2\phi = \cos \theta + i \sin \theta ;$$

hence, by Theorem 12-2a,

$$2\phi = \theta + 2k\pi , \text{ for some integer } k ,$$

or

$$\phi = \tfrac{1}{2}\theta + k\pi , \text{ for some integer } k .$$

Now z was given, and if $\theta = \arg z$ then we know θ too. Moreover, $0 \leq \theta < 2\pi$. If we suppose $\phi = \arg w$, the restriction $0 \leq \phi < 2\pi$ limits the possible values of the integer k . Indeed k can only be either 0 or 1 ; for, with $0 \leq \theta < 2\pi$, we have $0 \leq \frac{1}{2}\theta < \pi$ and if

$$k = 0: \quad \phi = \tfrac{1}{2}\theta + 0 \cdot \pi , \text{ so } 0 \leq \phi = \tfrac{1}{2}\theta < \pi < 2\pi .$$

$$k = 1: \quad \phi = \tfrac{1}{2}\theta + 1 \cdot \pi , \text{ so } 0 \leq \phi = \tfrac{1}{2}\theta + \pi < \pi + \pi = 2\pi;$$

but if

$$k \leq -1: \phi = \tfrac{1}{2}\theta + k\pi \leq \tfrac{1}{2}\theta - \pi < \pi - \pi = 0 ,$$

$$k \geq 2: \phi = \tfrac{1}{2}\theta + k\pi \geq \tfrac{1}{2}\theta + 2\pi \geq 2\pi .$$

We therefore find precisely two candidates for elements of the solution set of the equation

$$w^2 = z , \quad z \text{ given, not zero.}$$

They are $w_0 = \sqrt{|z|}\,[\cos(\frac{1}{2}\theta + 0 \cdot \pi) + i \sin(\frac{1}{2}\theta + 0 \cdot \pi)]$

$$w_1 = \sqrt{|z|}\,[\cos(\tfrac{1}{2}\theta + 1 \cdot \pi) + i \sin(\tfrac{1}{2}\theta + 1 \cdot \pi)] .$$

The question still remains whether or not the solution set is empty. As a matter of fact it is not, and both of our candidates are members of it. To see this we have to show that they satisfy the equation. We use de Moivre's theorem:

$$\begin{aligned} w_0^2 &= [\sqrt{|z|}\,[\cos(\tfrac{1}{2}\theta + 0 \cdot \pi) + i \sin(\tfrac{1}{2}\theta + 0 \cdot \pi)]]^2 \\ &= |z|\,(\cos(\theta + 0 \cdot 2\pi) + i \sin(\theta + 0 \cdot 2\pi)) \\ &= |z|\,(\cos\theta + i \sin\theta) \\ &= z ; \end{aligned}$$

and
$$
\begin{aligned}
w_1^{\,2} &= \{\sqrt{|z|}\,[\cos(\tfrac{1}{2}\theta + 1 \cdot \pi) + i \sin(\tfrac{1}{2}\theta + 1 \cdot \pi)]\}^2 \\
&= |z|\,\{\cos(\theta + 1 \cdot 2\pi) + i \sin(\theta + 1 \cdot 2\pi)\} \\
&= |z|\,(\cos\theta + i \sin\theta) \\
&= z \ .
\end{aligned}
$$

These conclusions are summarized in the following theorem.

Theorem 12-4a: The solution set of the equation

$$w^2 = z \ ,$$

where z is a given non-zero complex number, is

$$\{w_0 \ , \ w_1\} \ ,$$

where

$$
\begin{aligned}
w_0 &= \sqrt{|z|}\,\{\cos(\tfrac{1}{2}\theta + 0 \cdot \pi) + i \sin(\tfrac{1}{2}\theta + 0 \cdot \pi)\} \\
w_1 &= \sqrt{|z|}\,\{\cos(\tfrac{1}{2}\theta + 1 \cdot \pi) + i \sin(\tfrac{1}{2}\theta + 1 \cdot \pi)\}
\end{aligned}
$$

and $\theta = \arg z$.

Three observations:

1. If z happens to be real, this theorem agrees with the results in Chapters 1 and 5. If $z > 0$, then $|z| = z$ and $\arg z = 0$, so that

$$
\begin{aligned}
w_0 &= \sqrt{|z|}\,[\cos(0 + 0 \cdot \pi) + i \sin(0 + 0 \cdot \pi)] = \sqrt{z} \\
w_1 &= \sqrt{|z|}\,[\cos(0 + 1 \cdot \pi) + i \sin(0 + 1 \cdot \pi)] = -\sqrt{z} \ .
\end{aligned}
$$

If $z < 0$, then $|z| = -z$ and $\arg z = \pi$, so that

$$
\begin{aligned}
w_0 &= \sqrt{|z|}\,[\cos(\tfrac{1}{2}\pi + 0 \cdot \pi) + i \sin(\tfrac{1}{2}\pi + 0 \cdot \pi)] = \sqrt{|z|}\,(i) = \sqrt{z} \\
w_1 &= \sqrt{|z|}\,[\cos(\tfrac{1}{2}\pi + 1 \cdot \pi) + i \sin(\tfrac{1}{2}\pi + 1 \cdot \pi)] = \sqrt{-z}\,(-i) = -\sqrt{z}
\end{aligned}
$$

2. The roots w_0 , w_1 are additive inverses of each other:

$$w_1 = \sqrt{|z|}\,[\cos(\tfrac{1}{2}\,\theta + \pi) + i\,\sin(\tfrac{1}{2}\,\theta + \pi)]$$

$$= \sqrt{|z|}\,[-\cos(\tfrac{1}{2}\theta) - i\,\sin(\tfrac{1}{2}\theta)]$$

$$= -\,w_0\,.$$

(However, it would not be correct to say that one of them has to be negative. Why?)

3. If $z = 0$, the solution set contains only one element. That element is 0 , for $w^2 = 0$ if and only if $w = 0$.

Example 12-4a: Find the square roots of i .

Solution: Since $|i| = 1$ and $\arg i = \frac{\pi}{2}$, the theorem give

$$w_0 = \cos(\tfrac{\pi}{4} + 0 \cdot \pi) + i\,\sin(\tfrac{\pi}{4} + 0 \cdot \pi) = \frac{1 + i}{\sqrt{2}}\ ,$$

$$w_1 = \cos(\tfrac{\pi}{4} + 1\,.\,\pi) + i\,\sin(\tfrac{\pi}{4} + 1 \cdot \pi) = \frac{-1 - i}{\sqrt{2}}\ .$$

Check:

$$\left(\frac{1 + i}{\sqrt{2}}\right)^2 = \frac{1 + 2i + i^2}{2} = \frac{2i}{2} = i$$

$$\left(\frac{-1 - i}{\sqrt{2}}\right)^2 = (-1)^2\left(\frac{1 + i}{\sqrt{2}}\right)^2 = 1 \cdot i = i\ .$$

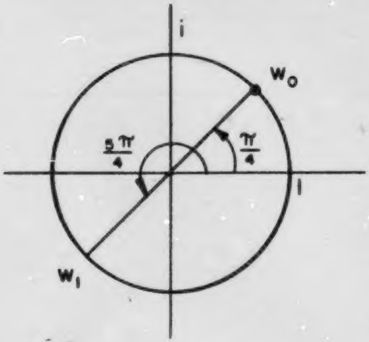


Fig. 12-4a

Note that $2 \arg w_0 = 2(\frac{\pi}{4}) = \frac{\pi}{2} = \arg i$,

$$2 \arg w_1 = 2(\frac{5\pi}{4}) = \frac{5\pi}{2} = 2\pi + \frac{\pi}{2} = 2\pi + \arg i \ .$$

Example 12-4b: Find the square roots of $\frac{12 + 5i}{2}$.

Solution: Let

$z = \frac{12 + 5i}{2}$; then $|z| = \frac{13}{2}$ and the polar form of z is

$$\frac{13}{2}(\frac{12 + 5i}{13}) = \frac{13}{2}(\cos \theta + i \sin \theta) \ .$$

Since $\cos \theta = \frac{12}{13} > 0$ and $\sin \theta = \frac{5}{13} > 0$, θ measures an angle in the first quadrant. We may get an estimate of θ by consulting a table of cosines or sines. Dividing this estimate by two we would have an estimate for the argument of w_0 . Re-entering the table we could get estimates for $\cos \frac{\theta}{2}$ and $\sin \frac{\theta}{2}$ and using them we could obtain an approximation to w_0 and hence also an approximation to $w_1 = - w_0$. The fact of the matter is that we need not settle for such approximations to square roots. We can calculate them exactly! For this purpose we use the "half-angle" formulas of Chapter 10.

$$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}} , \ \sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}} \ .$$

(Recall that the choice of signs in these formulas is determined by the quadrant containing $\frac{\theta}{2}$.) Returning to our example $\theta = \arg \frac{12 + 5i}{2}$ lies in the first quadrant; hence $\frac{\theta}{2}$ is also in the first quadrant. Thus $\cos \frac{\theta}{2}$ and $\sin \frac{\theta}{2}$ are both positive. We get

$$\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 + \frac{12}{13}}{2}} = \sqrt{\frac{25}{26}} = \frac{5}{\sqrt{26}} ,$$

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - \frac{12}{13}}{2}} = \sqrt{\frac{1}{26}} = \frac{1}{\sqrt{26}} \ .$$

Hence

$$\begin{aligned} w_0 &= \sqrt{\tfrac{13}{2}}\,(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}) \\ &= \sqrt{\tfrac{13}{2}}\,(\tfrac{5 + i}{\sqrt{26}}) \quad \text{in polar form,} \\ &= \tfrac{5}{2} + \tfrac{i}{2} \quad \text{in standard form.} \end{aligned}$$

The two square roots of $\frac{12 + 5i}{2}$ are therefore $\frac{5 + i}{2}$ and $-\frac{5 + i}{2}$.

Exercises 12-4

In each of Exercises 1, ... , 6 find the square roots of the given complex number z . Check your answers by squaring. Plot z and its square roots in an Argand diagram.

1. $z = 4$.
2. $z = 2 + 2i$.
3. $z = -9$.
4. $z = \sqrt{3} - i$.
5. $z = 3 + 4i$.
6. $z = \sqrt{2} + i\sqrt{3}$.
7. Let $w_k = \sqrt{|z|}\,[\cos(\frac{1}{2}\theta + k\pi) + i\sin(\frac{1}{2}\theta + k\pi)]$. Show that $w_{2k} = w_0$ and $w_{2k+1} = w_1$ for any natural number k .

12-5. Quadratic Equations with Complex Coefficients.

We announced in Chapter 5, Section 5-9, that each quadratic equation with complex coefficients has complex roots. In this section we prove that this is the case.

Consider the equation

$$Az^2 + Bz + C = 0 , \tag{12-5a}$$

where A, B, C are given complex numbers (some or all of which may be real), and $A \neq 0$. Completing the square, we have

$$(z + \frac{B}{2A})^2 = \frac{B^2 - 4AC}{4A^2} . \tag{12-5b}$$

In Chapter 5, where A, B, C were real, the right-hand member of Equation 12-5b was a real number. In the present case, it is a complex number (perhaps real, perhaps not). Let us write

$$E = \frac{B^2 - 4AC}{4A^2}$$

and

$$w = z + \frac{B}{2A} .$$

Then Equation 12-5b becomes

12-5c $$w^2 = E .$$

There are two cases: (1) $E = 0$, and (2) $E \neq 0$. If $E = 0$, which means $B^2 - 4AC = 0$, the solution set of Equation 12-5c is the set $\{0\}$. Since $z = -\frac{B}{2A} + w$, the solution set of Equations 12-5a and 12-5b is the set $\{-\frac{B}{2A}\}$. Thus in Case (1), $B^2 - 4AC = 0$, Equation 12-5a has just one solution, $-\frac{B}{2A}$.

In Case (2), $B^2 - 4AC \neq 0$, Equation 12-5c has two solutions, say w_0 and w_1 . We know, however, from Section 12-4, that $w_1 = -w_0$. Thus we may write $\{w_0 , -w_0\}$ for the solution set of Equation 12-5c. The solution set of Equation 12-5a is then

$$\{-\frac{B}{2A} + w_0 , \ -\frac{B}{2A} - w_0\} ,$$

w_0 being one of the solutions of

$$w^2 = E = \frac{B^2 - 4AC}{4A^2} .$$

We state these results as a theorem.

Theorem 12-5a: The solution set of the equation

$$Az^2 + Bz + C = 0\ , \quad A \neq 0\ ,$$

is $\{-\frac{B}{2A}\}$ if $B^2 - 4AC = 0$; if $B^2 - 4AC \neq 0$, the solution set is

$$\{-\frac{B}{2A} + w_0\ ,\quad -\frac{B}{2A} - w_0\}\ ,$$

where w_0 is either of the solutions of the equation

$$w^2 = \frac{B^2 - 4AC}{4A^2}\ .$$

Example 12-5a: Solve $z^2 - (2 + 4i)z + 4i = 0$.

Solution: Here

$A = 1$, $B = -(2 + 4i)$, $C = 4i$, $B^2 - 4AC = 4(1 + 2i)^2 - 16i = -12$, $E = -3$. Let $w_0 = i\sqrt{3}$; then

$$z_1 = -\frac{B}{2A} + w_0 = 1 + 2i + i\sqrt{3} = 1 + (2 + \sqrt{3})i$$

$$z_2 = -\frac{B}{2A} - w_0 = 1 + 2i - i\sqrt{3} = 1 + (2 - \sqrt{3})i\ .$$

Example 12-5b: Solve $z^2 + (1 + i)z + i = 0$.

Solution: Here

$A = 1$, $B = 1 + i$, $B^2 - 4AC = (1 + i)^2 - 4i = 2i - 4i = -2i$, $E = -\frac{i}{2} = \frac{1}{2}(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$.

Hence $w_0 = \frac{1}{\sqrt{2}}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = -\frac{1 + i}{2}$, and

$$z_1 = -\frac{B}{2A} + w_0 = -\frac{1 + i}{2} + \frac{1 - i}{2} = -i$$

$$z_2 = -\frac{B}{2A} - w_0 = -\frac{1 + i}{2} - \frac{1 - i}{2} = -1\ .$$

Example 12-5c: Solve the equation

$$z^2 + (1 - 5i)z - (12 + 5i) = 0\ .$$

Solution: Here

$A = 1$, $B = 1 - 5i$, $C = -12 - 5i$, $B^2 - 4AC = (1 - 5i)^2 + 4(12 + 5i) = 24 + 10i$, $E = \frac{12 + 5i}{2}$. In Example 12-4b, we found that the solutions of $w^2 = \frac{12 + 5i}{2}$ are $\frac{5 + i}{2}$, $-\frac{5 + i}{2}$. Taking $w_0 = \frac{5 + i}{2}$, we have

$$z_1 = -\frac{B}{2A} + w_0 = \frac{-1 + 5i}{2} + \frac{5 + i}{2} = 2 + 3i ,$$

$$z_2 = -\frac{B}{2A} - w_0 = \frac{-1 + 5i}{2} - \frac{5 + i}{2} = -3 + 2i .$$

(Compare with Section 5-9 of Chapter 5, where this quadratic equation was mentioned.)

Exercises 12-5

Solve the following equations:

1. $z^2 - iz + 2 = 0$.
2. $iz^2 + (i - 1)z - 1 = 0$.
3. $z^2 - 2iz - 1 = 0$.
4. $z^2 - (2 + 2i)z + 2i = 0$.
5. $z^3 + 2iz^2 + 3iz = 0$.
6. $z^4 + 4\sqrt{2}\, iz^2 - 8 = 0$.

*7. $z^4 - iz^2 + 1 - 3i = 0$. (Use half-angle formulas to obtain z^2, tables to get approximations for z .)

8. $(z^3 - iz^2) - (1 + 2i)(z^2 - iz) - (iz + 1) = 0$.

12-6. Roots of Order n.

In Section 12-4 we discussed the solution of the equation $w^2 = z$, where z is a given complex number. In this section we consider the equation $w^n = z$, where z is a given complex number and n is a natural number. First we consider the case $n = 3$, and later we extend our results to an arbitrary natural number n .

For $z = 0$, the equation $w^n = z$ has only one solution, which is 0. (Why?) We shall find that there are n distinct roots when $z \neq 0$.

Cube roots.

Consider the equation

$$w^3 = z , \tag{12-6a}$$

where z is given. Suppose $z \neq 0$. We proceed as we did in Section 12-4, when we discussed the equation $w^2 = z$. If

$$w = |w| (\cos \phi + i \sin \phi)$$

is in the solution set of Equation 12-6a, then

$$|w|^3 (\cos 3\phi + i \sin 3\phi) = |z| (\cos \theta + i \sin \theta)$$

so $|w|^3 = |z|$, $\cos 3\phi + i \sin 3\phi = \cos \theta + i \sin \theta$;

and $|w| = \sqrt[3]{|z|}$, $3\phi = \theta + k \cdot 2\pi$, for some integer k,

or $|w| = \sqrt[3]{|z|}$, $\phi = \frac{1}{3}\theta + k \cdot \frac{2\pi}{3}$, $k \in I$.

Note that $\sqrt[3]{|z|}$ is the (real) cube root of the positive real number $|z|$; it is therefore positive. We propose to show that, if $0 \leq \theta < 2\pi$, we have $0 \leq \phi < 2\pi$ if and only if $k = 0, 1, 2$. (Compare this to the analogous situation in Section 12-4.) Indeed, for

$k = 0$: $\phi = \frac{1}{3}\theta$, so $0 \leq \phi < \frac{2\pi}{3} < 2\pi$,

$k = 1$: $\phi = \frac{1}{3}\theta + \frac{2\pi}{3}$, so $0 \leq \phi < \frac{2}{3}\pi + \frac{2}{3}\pi = \frac{4}{3}\pi < 2\pi$,

$k = 2$: $\phi = \frac{1}{3}\theta + \frac{4\pi}{3}$, so $0 < \phi < \frac{2}{3}\pi + \frac{4\pi}{3} = 2\pi$.

But if $\quad k \geq 3: \ \phi = \frac{1}{3}\theta + \frac{2k\pi}{3} \geq 0 + \frac{6\pi}{3} = 2\pi$,

and if $\quad k \leq -1: \ \phi = \frac{1}{3}\theta + \frac{2k\pi}{3} < \frac{2\pi}{3} - \frac{2\pi}{3} = 0$.

As with square roots, de Moivre's theorem shows that each of the numbers

$$\text{12-6b} \quad \begin{cases} w_0 = \sqrt[3]{|z|}\,[\cos(\frac{1}{3}\theta + 0 \cdot \frac{2\pi}{3}) + i\sin(\frac{1}{3}\theta + 0 \cdot \frac{2\pi}{3})] \\ w_1 = \sqrt[3]{|z|}\,[\cos(\frac{1}{3}\theta + 1 \cdot \frac{2\pi}{3}) + i\sin(\frac{1}{3}\theta + 1 \cdot \frac{2\pi}{3})] \\ w_2 = \sqrt[3]{|z|}\,[\cos(\frac{1}{3}\theta + 2 \cdot \frac{2\pi}{3}) + i\sin(\frac{1}{3}\theta + 2 \cdot \frac{2\pi}{3})] \end{cases}$$

really is a member of the solution set. We summarize these results as a theorem.

Theorem 12-6a: The solution set of the equation

$$w^3 = z ,$$

where z is a given non-zero complex number, is

$$\{w_0 , w_1 , w_2\} ,$$

where

$$w_0 = \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta + 0 \cdot \tfrac{2\pi}{3}) + i\sin(\tfrac{1}{3}\theta + 0 \cdot \tfrac{2\pi}{3})]$$

$$w_1 = \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta + 1 \cdot \tfrac{2\pi}{3}) + i\sin(\tfrac{1}{3}\theta + 1 \cdot \tfrac{2\pi}{3})]$$

$$w_2 = \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta + 2 \cdot \tfrac{2\pi}{3}) + i\sin(\tfrac{1}{3}\theta + 2 \cdot \tfrac{2\pi}{3})]$$

and $\theta = \arg z$.

Example 12-6a: Find the cube roots of $1 = 1 + 0 \cdot i$.

Solution: Here $|z| = 1$ and $\arg z = 0$. Our formulas give

$$\begin{aligned} w_0 &= \cos(0 + 0 \cdot \tfrac{2\pi}{3}) + i \sin(0 + 0 \cdot \tfrac{2\pi}{3}) \\ &= \cos 0 + i \sin 0 = 1 \; ; \\ w_1 &= \cos(0 + 1 \cdot \tfrac{2\pi}{3}) + i \sin(0 + 1 \cdot \tfrac{2\pi}{3}) \\ &= \cos \tfrac{2\pi}{3} + i \sin \tfrac{2\pi}{3} = \frac{-1 + i\sqrt{3}}{2}; \\ w_2 &= \cos(0 + 2 \cdot \tfrac{2\pi}{3}) + i \sin(0 + 2 \cdot \tfrac{2\pi}{3}) \\ &= \cos \tfrac{4\pi}{3} + i \sin \tfrac{4\pi}{3} = \frac{-1 - i\sqrt{3}}{2} \, . \end{aligned}$$

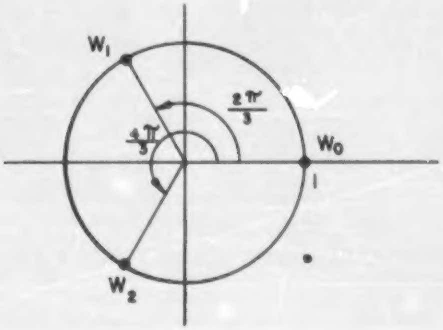


Fig. 12-6a

Check: $1^3 = 1$,

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = \frac{1}{8}\left[(-1)^3 + 3(\pm i\sqrt{3}) - 3(\pm i\sqrt{3})^2 + (\pm i\sqrt{3})^3\right]$$

$$= \frac{1}{8}\left[-1 + 9 + i(\pm 3\sqrt{3} \mp 3\sqrt{3})\right] = 1 \, .$$

There is a very important connection between the results obtained in Example 12-6a and the Formula 12-6b. Let us give names to the special numbers

$$1, \ \frac{-1 + i\sqrt{3}}{2}, \ \frac{-1 - i\sqrt{3}}{2},$$

the three cube roots of 1. If we put

$$\omega = \frac{-1 + i\sqrt{3}}{2},$$

then $$\omega^2 = \left(\frac{-1 + i\sqrt{3}}{2}\right)^2 = \frac{1}{4}(1 - 2i\sqrt{3} - 3)$$
$$= \frac{1}{4}(-2 - 2i\sqrt{3}) = \frac{-1 - i\sqrt{3}}{2} ,$$

and ω^2 is the other non-real complex cube root of 1 . The three cube roots of 1 are, therefore, ω , ω^2 , ω^3 , since $\omega^3 = 1$. The connection between the cube roots of 1 and the cube roots of any complex number z is given in the following theorem.

Theorem 12-6b: If z is not zero and w is any one solution of the equation $w^3 = z$, then the other two solutions are ωw and $\omega^2 w$.

We give two proofs of Theorem 12-6b. The first proof, which involves less computation, accomplishes all that is actually required. The second proof exhibits explicitly the relationship between the Formulas 12-6b and the much more compact expressions w, ωw, $\omega^2 w$.

First Proof. Our first assertion is that the three numbers 1 , ω , ω^2 are distinct. This is evident on the grounds that no pair of them have the same real and imaginary parts. Moreover, it is impossible for any two of the numbers w , ωw , $\omega^2 w$ to be equal if $w^3 = z \neq 0$. For, on the contrary, we should have

$$w = \omega w$$

(say) or $1 = \omega$ since w cannot be zero. This contradicts the fact that 1 and ω are distinct. We know then, that w , ωw , and $\omega^2 w$ are three different numbers. We propose to prove that each of these numbers satisfies the equation $w^3 = z$. Since they are three different numbers they must be the three elements of the solution set, for that set contains only three elements altogether.

By hypothesis $w^3 = z$. Now, in addition,

$$(\omega w)^3 = \omega^3 w^3 = w^3 = z \text{ , for } \omega^3 = 1 \text{ ,}$$

and $$(\omega^2 w)^3 = \omega^6 w^3 = 1^2 w^3 = w^3 = z \text{ ,}$$

so ωw and $\omega^2 w$ also satisfy the equation.

Second Proof: We show that

$$\omega w_0 = w_1 \text{ and } \omega^2 w_0 = w_2 \text{ ,}$$

and leave the other possibilities as an exercise for the student. Recall that

$$w_0 = \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta) + i \sin(\tfrac{1}{3}\theta)] \text{ ,}$$

$$\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \text{ , } \omega^2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \text{ .}$$

We have, then,

$$\begin{aligned} \omega w_0 &= \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta) + i \sin(\tfrac{1}{3}\theta)]\,(\cos \tfrac{2\pi}{3} + i \sin \tfrac{2\pi}{3}) \\ &= \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta + \tfrac{2\pi}{3}) + i \sin(\tfrac{1}{3}\theta + \tfrac{2\pi}{3})] \\ &= w_1 \text{ ;} \end{aligned}$$

$$\begin{aligned} \omega^2 w_0 &= \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta) + i \sin(\tfrac{1}{3}\theta)]\,(\cos \tfrac{4\pi}{3} + i \sin \tfrac{4\pi}{3}) \\ &= \sqrt[3]{|z|}\,[\cos(\tfrac{1}{3}\theta + \tfrac{4\pi}{3}) + i \sin(\tfrac{1}{3}\theta + \tfrac{4\pi}{3})] \\ &= w_2 \text{ .} \end{aligned}$$

This theorem tells us a great deal about the geometry of cube roots. It is the analogue for $n = 3$ of the fact that $w_1 = -w_0$ when $n = 2$. We know that the cube roots of z lie on the circle with center at 0 and radius $\sqrt[3]{|z|}$. One of them, the one we call w_0 , has argument one-third of the argument of z .

Since the other two are ωw_0 and $\omega^2 w_0$, we can see at once that the three cube roots are equally spaced around this circle. This is clear when $z = 1$ and the roots are $1, \omega, \omega^2$. For any other value of z we have merely to add $\frac{1}{3}\theta$, or $\frac{1}{3}\arg z$, to the arguments of $1, \omega, \omega^2$ to get the arguments of w_0, w_1, w_2, respectively. Since we add the same quantity to each of these arguments we turn the whole configuration around 0 by the amount added.

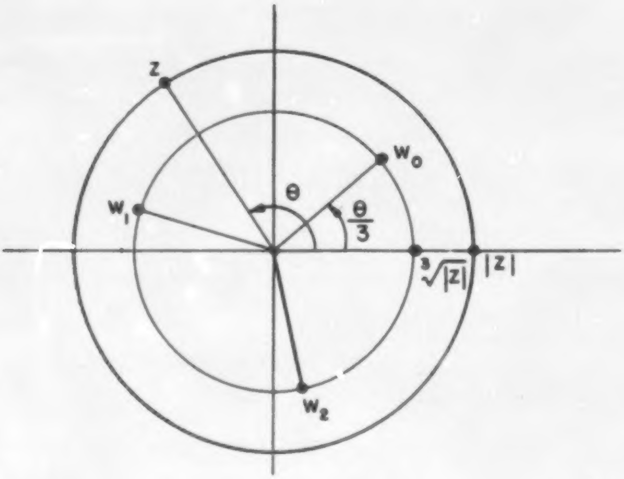


Fig. 12-6b

Roots of order n .

We now extend the results obtained for square roots and cube roots. We give theorems for the roots of the equation

12-6c $$w^n = z \; , \; z \neq 0 \; ,$$

where n is any natural number. The student will note that substituting 2 and 3 for n in the theorems and their proofs gives us the theorems obtained for square roots and cube roots as well as their proofs. Thus there is no new idea in the remainder of this section; we merely carry over what we did before to the general case.

Theorem 12-6c: If n is a natural number and z is a given non-zero complex number, the solution set of the equation

$$w^n = z ,$$

is $\{w_0 , w_1 , \dots , w_{n-1}\}$, where

$$w_k = \sqrt[n]{|z|}\,[\cos(\tfrac{1}{n}\theta + k \cdot \tfrac{2\pi}{n}) + i \sin(\tfrac{1}{n}\theta + k \cdot \tfrac{2\pi}{n})] ,$$

$$k = 0 , 1 , \dots , n - 1 , \text{ and } \theta = \arg z .$$

Proof: By de Moivre's theorem, each of the numbers w_k , $k = 0 , 1 , \dots , n - 1$, belongs to the solution set since

$$\begin{aligned} w_k{}^n &= \{\sqrt[n]{|z|}\,[\cos(\tfrac{1}{n}\theta + k \cdot \tfrac{2\pi}{n}) + i \sin(\tfrac{1}{n}\theta + k \cdot \tfrac{2\pi}{n})]\}^n \\ &= |z|\,[\cos(\theta + k \cdot 2\pi) + i \sin(\theta + k \cdot 2\pi)] \\ &= |z|\,(\cos\theta + i \sin\theta) \\ &= z . \end{aligned}$$

Moreover, they are all distinct for no two have the same argument. On the other hand, suppose w belongs to the solution set and that $\phi = \arg w$. We must show that each element of the solution set is one of the numbers $w_0 , w_1 , \dots , w_{n-1}$. We assume, then, that $w^n = z$. This implies

$$|w|^n = |z| \text{ and } n\phi = \theta + k \cdot 2\pi ,$$

for some integer k . Thus

$$|w| = \sqrt[n]{|z|} \text{ and } \phi = \tfrac{1}{n}\theta + k \cdot \tfrac{2\pi}{n} .$$

Now $0 \leq \theta < 2\pi$ and $0 \leq k \leq n - 1$ give

$$0 < \phi = \frac{1}{n}\theta + k \cdot \frac{2\pi}{n} < \frac{2\pi}{n} + k \cdot \frac{2\pi}{n} = (k + 1)\frac{2\pi}{n} \leq n \cdot \frac{2\pi}{n} = 2\pi;$$

while $0 \leq \theta < 2\pi$ and $k \geq n$ give

$$\phi = \frac{1}{n}\theta + k \cdot \frac{2\pi}{n} \geq 0 + n \cdot \frac{2\pi}{n} \geq 2\pi ,$$

and $0 \leq \theta < 2\pi$ and $k \leq -1$ give

$$\phi = \frac{1}{n}\theta + k \cdot \frac{2\pi}{n} < \frac{2\pi}{n} - \frac{2\pi}{n} = 0 .$$

This theorem shows that each non-zero complex number z has n distinct complex n^{th} roots, where n is any natural number. The complex number 0 has only one n^{th} root for each natural number n. It is 0.

The n complex n^{th} roots of the number 1 are called the n^{th} roots of unity. Let

$$\omega = \cos(\frac{1}{n}0 + 1 \cdot \frac{2\pi}{n}) + i \sin(\frac{1}{n}0 + 1 \cdot \frac{2\pi}{n})$$

$$= \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n}) ,$$

so that ω is a particular one of the n^{th} roots of unity. De Moivre's theorem shows us that the $n - 1$ other n^{th} roots of unity are

$$\omega^2 , \omega^3 , \ldots , \omega^n ,$$

since

$$\omega^k = [\cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})]^k$$

$$= \cos(k \cdot \frac{2\pi}{n}) + i \sin(k \cdot \frac{2\pi}{n}) ,$$

thus for $k = 1 , 2 , 3 , \ldots , n$ we obtain precisely the same roots given by Theorem 12-6c on putting $z = 1$, $\theta = 0$.

The next theorem generalizes a result we found in the cases $n = 2, 3$.

Theorem 12-6d: If w is any one of the roots of the equation $w^n = z$, where $z \neq 0$, then the solution set of the equation may be described as

$$\{w, \omega w, \omega^2 w, \ldots, \omega^{n-1} w\},$$

where $\omega = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$.

Proof: No two of the numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ can be equal because their arguments are, respectively, 0, $\frac{2\pi}{n}, 2 \cdot \frac{2\pi}{n}, 3 \cdot \frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$ and no two of these arguments are equal. Hence no two of the numbers

12-6d $\qquad 1 \cdot w, \omega \cdot w, \omega^2 w, \ldots, \omega^{n-1} w$

can be equal; for otherwise w would have to be zero, which is impossible. We know, by Theorem 12-6c, that Equation 12-6c has exactly n roots; we complete the proof by showing that each of the numbers 12-6d is a solution of $w^n = z$. But this is easy, since

$$(\omega^k w)^n = (\omega^n)^k w^n = 1^k z = z,$$

for any integer k.

This theorem extends to the general case the results we found for cases $n = 2, 3$ on the location of the roots. All the roots lie on the circle with center 0 and radius $\sqrt[n]{|z|}$. If $\theta = \arg z$, one of these roots has argument $\frac{1}{n}\theta$; the other roots are located at equal distances around the circle. For $n > 2$, the n roots, therefore, represent the vertices of a regular polygon of n sides inscribed in the circle. It therefore suffices to locate one of them--say w_0; after this the positions of all the others are determined.

Exercises 12-6

In each of Exercises 1, ... , 7 find the cube roots of the given complex numbers.

1. $z = 2$.
2. $z = -2$.
3. $z = i$.
4. $z = -i$.
5. $z = i - 1$.
6. $z = 3 + 4i$. (Use tables to obtain approximations to the cube roots.)

*7. $z = 1 + i$. (Do not use tables.)

8. Solve the equations
   (a) $x^4 = -1$.
   (b) $x^6 - 1 = 0$.
   (c) $x^3 + (6 + 6\sqrt{3}\,i) = 0$.
9. Using tables find the 4th roots of
   $$16(\cos 164^\circ + i \sin 164^\circ) .$$
10. Show that the sum of the n n^{th} roots of unity is zero.

*11. Find n complex roots of each of the following equations:
   (a) $z^n + z^{n-1} + z^{n-2} + \dots + z^3 + z^2 + z + 1 = 0$, where n is a natural number;
   (b) $z^n - z^{n-1} + z^{n-2} - \dots - z^3 + z^2 - z + 1 = 0$, where n is an even natural number.

Section 12-7. Miscellaneous Exercises

1. Prove that $\omega_1 = \omega_2^2$ and $\omega_2 = \omega_1^2$, where ω_1 and ω_2 are the two non-real cube roots of unity.
2. Express in polar form:
   (a) $-3 + \sqrt{3}\,i$.
   (b) $-2 - 2i$.
   (c) $\cos 217^\circ - i \sin 217^\circ$.
   (d) $0.5592 - 0.8290\,i$.

(e) The conjugate of $r(\cos\theta + i\sin\theta)$.

(f) $7(-\cos 25^\circ 35' + i\sin 25^\circ 35')$.

(g) $\cos 182^\circ + i\sin 358^\circ$.

(h) $\cos 23^\circ + i\sin 32^\circ$. (Use tables.)

3. Express in polar form and perform the indicated operations:

(a) $(-1 + i)(1 - \sqrt{3}\,i)$.

(b) $\dfrac{3 - \sqrt{3}\,i}{5 - 5i}$.

(c) $\dfrac{(\cos 137^\circ + i\sin 763^\circ)(\cos 317^\circ + i\sin 223^\circ)}{\cos(-30^\circ) - i\sin 330^\circ}$.

(d) $\dfrac{(1 - i)^2}{(1 - \sqrt{3}\,i)^5}$.

(e) $(\cos 10^\circ + i\sin 15^\circ)(\cos 15^\circ - i\sin 10^\circ)$.

4. Simplify the product $1 \cdot \omega \cdot \omega^2 \cdot \omega^3 \cdot \cdot \cdot \omega^{n-1}$,

(a) when n is even.

(b) when n is odd.

5. Let z be a complex number and ω a non-real cube root of unity. Show that the points z , ωz , $\omega^2 z$ form an equilateral triangle on the Argand diagram.

6. Express as a function of z and n the length of one side of a regular n-sided polygon inscribed in a circle of radius $|z|$, where z is a complex number.

7. Find all the roots of each of the following equations:

(a) $x^4 - 2 - 2i = 0$.

(b) $8z^6 + \frac{z}{4} = 0$.

APPENDIX

*12-7. The Functions $z^{\frac{1}{2}}$, $z^{\frac{1}{3}}$, ... , $z^{\frac{1}{n}}$.

We have seen that if z is any non-zero complex number, $\theta = \arg z$, and m is any integer, then

$$z^m = |z|^m [\cos(m\theta) + i \sin(m\theta)] .$$

In Section 12-6 we studied the equation $w^n = z$, where z is a given non-zero complex number and n is a given natural number; we found that the solution set consists of the n distinct numbers

$$w_k = \sqrt[n]{|z|} [\cos(\frac{1}{n} \theta + k \cdot \frac{2\pi}{n}) + i \sin(\frac{1}{n} \theta + k \cdot \frac{2\pi}{n})] ,$$

$$k = 0 , 1 , 2 , 3 , \ldots , n - 1 .$$

The results in Chapter 9 tell us that if a is a positive real number, there is a unique positive real number b such that $b^n = a$. We write, in this case, $b = a^{1/n}$. Moreover $(a^{1/n})^m = (a^m)^{1/n}$, entitling us to write $a^{m/n}$ to denote either of these numbers. We also know from Chapter 9 that the familiar "laws" of integral exponents carry over entirely to these "fractional" exponents.

In this section we propose to consider the question of "fractional" powers of complex numbers. It should be apparent at the outset that our task is much more involved in the complex case than it was in the real case--if only because we have n roots of order n here instead of just one.

The first step in any study of rational powers is to give a meaning to expressions of the form $z^{1/n}$. We cannot simply say, as we could in the real case, that it is the solution of the equation $w^n = z$; for there are too many solutions (more than 1 if $n > 1$) . To put the matter rather bluntly, the relation $w^n = z$ does not define w as a function of z . In order to construct some sort of function in this context, we are therefore forced to shift our point of view. We have a correspondence here, but in order to obtain a function we must first settle the questions of what are its domain and range.

Let us take the simplest case first; we therefore consider the equation $w^2 = z$. We hope eventually to find how w can be considered to depend on z . In order to understand this relation, however, we shall first turn it around and investigate in some detail how $z = w^2$ depends on w . By shifting our attention to this more familiar situation we can learn much that will help us in discussing the more complicated "inverse" relation. We now have a function $z = w^2$ to work with. For our study of this function let us draw two pictures.

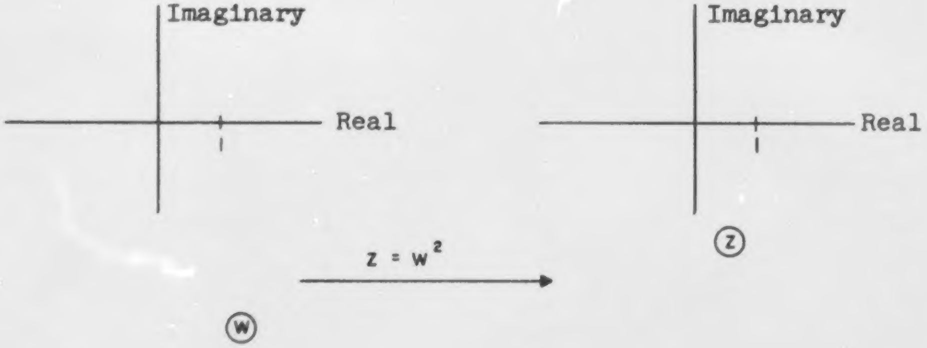


Fig. 12-7a

In the first we shall plot the complex number w and in the second we shall plot z. The functional relationship between these variables can be described by determining which points in our "z-plane" correspond to given points in our "w-plane."

Let us trace some of the pairs in this correspondence. For instance, if $w = 1$, the corresponding point in the z-plane is 1. If w is real and greater than 1 (on the "real" axis in the w-plane to the right of 1), so is its "image" in the z-plane; indeed it is further away from 1 since $|z| > |w|$ if $z = w^2$ and $|w| > 1$. The image of any point in the w-plane outside of the unit circle lies outside the unit circle in the z-plane for $|z| > |w| > 1$ if $|w| > 1$. Also each point in the w-plane and inside the unit circle corresponds to a point inside the unit circle of the z-plane. Finally each point on the unit circle of the w-plane corresponds to a point on the unit circle of the z-plane. Our discussion of the correspondence has taken into account only the absolute values of w and z so far, and may be considered the geometrical version of the statements:

$$\text{If } z = w^2, \text{ then } \begin{aligned} |z| &> |w| \quad \text{for} \quad |w| > 1, \\ |z| &= |w| \quad \text{for} \quad |w| = 1, \\ |z| &< |w| \quad \text{for} \quad |w| < 1. \end{aligned}$$

To complete our picture, we consider $\arg w$ and $\arg z$. We may do this by tracing the image in the z-plane of a point moving around the unit circle of the w-plane. The image point moves meantime on the unit circle of the z-plane. Note that as w makes the trip through the first quadrant on its unit circle, going from 1 to i, z manages to travel through both the first and second quadrants, going all the way from 1 through i and on to -1 (Figure 12-7b).

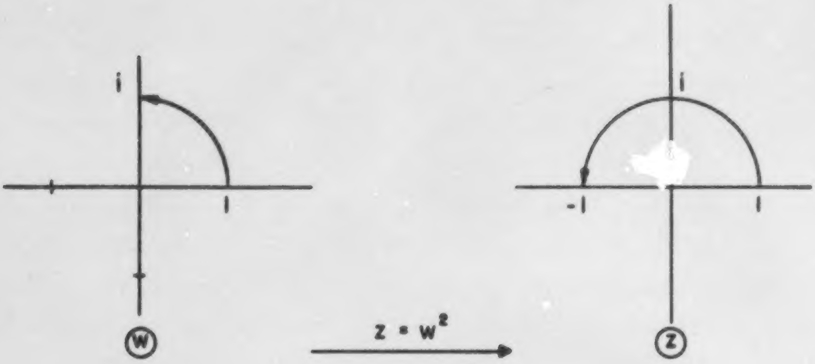


Fig. 12-7b

Let w continue, passing through the second quadrant on its unit circle. When this happens, z shoots on around its unit circle completing a full circuit (Figure 12-7c) .

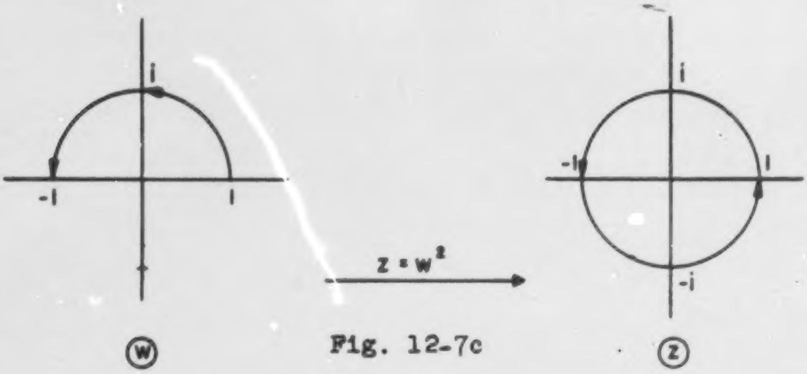


Fig. 12-7c

What happens next is a good question. It depends on how you choose to describe it. If w keeps going, passing through the third quadrant, z will shoot along through its first and second quadrants again. And finally as w passes through its fourth quadrant, z will race through its third and fourth quadrants; and, lapping w , they will come into their respective points 1 together. This is the trouble. If z went through each of the points on its unit circle exactly once as w makes its full circuit there would be no problem; there would be just one value of w corresponding to each of these values of z , and we would have a function $w = f(z)$ to talk about. As it is we have not yet got such a function since each z gives rise to a pair of w's .

We can get around this difficulty by a trick--at least it was a trick when it was introduced about a hundred years ago. But no trick can remain a trick for a hundred years--certainly not one as good as this. It has become quite a respectable method since it was introduced and has come to be considered one of the most important methods for treating questions of this sort.

Our trouble amounts to the fact that we have to use the points in the z-plane twice to describe a tour such as the one considered. Suppose then that we use two z "planes" going through each of them just once. Can we do this somehow? The famous German mathematician Bernhard Riemann found that we can, provided we are sufficiently ingenious about it. He visualized the "two" z "planes" arranged as follows: We "cut" each of them along the positive real axis and then "glue" them together in criss-cross fashion as shown (Figure 12-7d). The resulting configuration is an example of what we call a Riemann Surface.

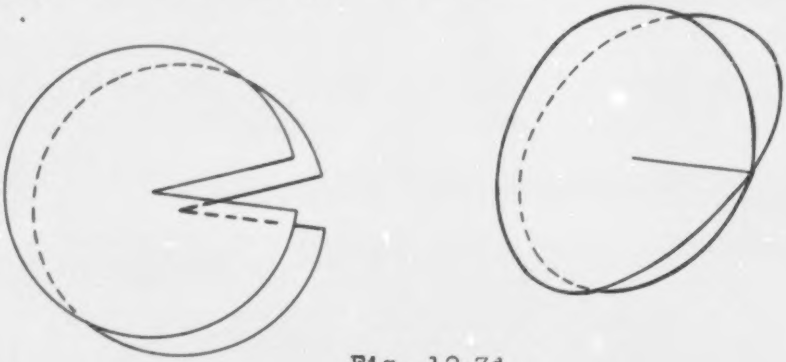


Fig. 12-7d

Our object in all this is to obtain w as a function of z if $w^2 = z$. Can we do it now? Again let w traverse its unit circle. This time, however, let us imagine z as moving on the Riemann Surface. Very well, when w is 1 , z is 1 . As w moves through the first quadrant on its unit circle we imagine z as moving halfway around its unit circle in one of the sheets of its Riemann Surface. As w passes through its second quadrant, z comes completely around and returns to 1 . Now--here is the trick--as w goes into its third quadrant, z will pass over to its other sheet and go through two quadrants of the unit circle on that sheet. (Remember the sheets cross each other along the positive real axis.) When w finishes its circuit, so does z . But, by introducing this way of looking at the matter z has gone through no point twice, except that it ends at 1 where it starts. This statement must be interpreted with care. There would appear to be a duplication since our "gluing" seems to identify the two points 1 of the two sheets. Let us imagine that $z = 1_1$ (in the first sheet) for $w = 1$, $z = 1_2$ (in the second sheet) for $w = -1$, and $z = 1_1$ when $w = 1$ again. We need all the points of each sheet--we cannot afford to throw any away by allowing some points to be in both sheets. We look on these points as distinct although it is hard to make a convincing drawing; the pieces are connected cross-wise but we think of them as not touching anywhere other than 0 .

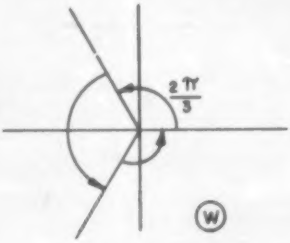
Hence, corresponding to each point on the unit "circle" in the Riemann Surface, there is one and only one point in the w-plane. Here is our function! Its domain is the two-sheeted Riemann Surface, its range is the w-plane. This function is denoted by $w = z^{1/2}$.

The same theme, with variations, runs through the discussions of $w^n = z$ for other natural number values of n .

Thus for $n = 3$, the function $z = w^3$ opens each of the "fans"

$0 = \phi < \frac{2\pi}{3}$, $\frac{2\pi}{3} \le \phi < \frac{4\pi}{3}$, $\frac{4\pi}{3} \le \phi < 2\pi$

in the sense the images in the $z = w^3$ plane of the points in each of them fill out the z-plane.



Thus the z-plane is covered three times by the images of points in the w-plane. In this case we replace the z-plane with a three-sheeted Riemann Surface shown in Figure 12-7e. As before, we then obtain a function $w = z^{1/3}$ whose domain is this surface and whose range is the w-plane.

The idea is analagous for a general n . The function $w = z^{1/n}$ has an n-sheeted Riemann Surface for its domain; its range, as before, is the w-plane.

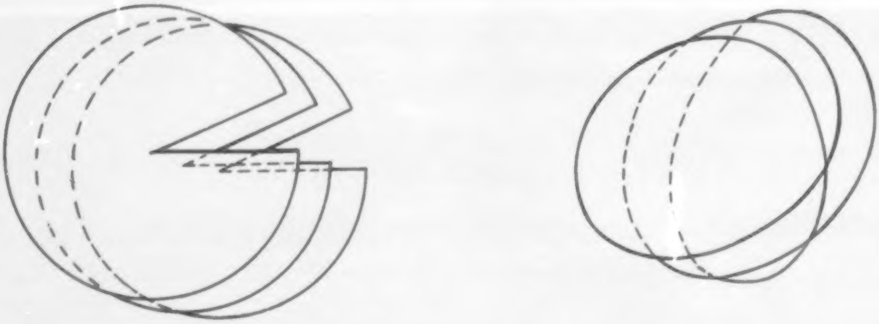


Fig. 12-7e

Chapter 13

SEQUENCES AND SERIES

13-1. Introduction.

It is a common experience to be confronted with a set of numbers arranged in some order. The order and arrangement may be given us, or we may have to discover a law for it from some data. For example, the milkman comes every other day. He came on July 17; will he come on August 12? We might consider that we are given the set of dates

$$17,\ 19,\ 21,\ \ldots$$

arranged from left to right in the order of increasing time. We wish to know how to continue the set. In this simple case the scheme is trivial; we have

$$17,\ 19,\ 21,\ \ldots,\ 29,\ 31,\ 2,\ 4,\ \ldots,\ 28,\ 30,\ \ldots,$$

and the answer to the original question is yes. Any such ordered arrangement of a set of numbers is called a sequence.

Definition 13-1a: A finite sequence of n terms is a function a whose domain is the set of numbers $\{1, 2, \ldots, n\}$. The range is then the set $\{a(1), a(2), \ldots, a(n)\}$, usually written $\{a_1, a_2, \ldots, a_n\}$. The elements of the range are called the terms of the sequence.

An infinite sequence is a function a whose domain is the set $\{1,2,3,\ldots,n,\ldots\}$ of all positive integers. The range of a is then the set $\{a(1), a(2), a(3), \ldots, a(n), \ldots\}$, usually written $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$. The element a_n of the range is called the n^{th} term of the sequence.

The terms of a finite or infinite sequence may be arbitrary objects of any kind, but in this course they will be real or complex numbers.

Example 13-1a:

(a) $1, 2, 3, \ldots, 17$
(b) $17, -23, 15, 5280$
(c) $17, 12, 7, 2, -3, -8$
(d) $3, 6, 9, 12, \ldots$
(e) $\pi, \pi^2, \pi^3, \ldots$
(f) $\sin \pi, \sin \frac{\pi}{2}, \sin \frac{\pi}{3}, \ldots, \sin \frac{\pi}{n}, \ldots$.

The first three sequences are finite; the last three are infinite. In all but (b) a definite law governing the formation of successive terms is easily discernible.

Suppose now that in the sequences above we replace the commas between successive terms with plus signs. The resulting expressions are called series. (The noun "series" is both singular and plural.)

Definition 13-1b: Let $\{a_1, a_2, \ldots, a_n\}$ be a given finite sequence of real or complex numbers; then the indicated sum

$$a_1 + a_2 + \ldots + a_n$$

is called a finite series. The numbers $a_1, a_2, \ldots, a_n$ are called the terms of the series.

Let $\{a_1, a_2, \ldots, a_n, \ldots\}$ be a given infinite sequence of real or complex numbers. Then the indicated sum

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an infinite series. The number a_n is called the n^{th} term of the infinite series.

According to the definition, the expression

$$1 + 4 + 7 + 10 + 13 + 16$$

is an example of a series. It is a finite series having six terms. Note that the operation of addition suggested by the plus signs is *not* actually involved in the definition. Of course, we shall eventually want to perform the addition in order to find the *sum of a series*, but it is wrong to confuse a *series* with its *sum*.

Example 13-1b:

$$1 + \frac{1}{10} + \frac{1}{10^2} + \ldots \quad .$$

This is an infinite series. Note the plus sign before the dots. The 10th term of the series is $\frac{1}{10^9} = 10^{-9}$. The student is warned against referring to the "last term" of an infinite series; there is none!

Example 13-1c: Find the 11th term of the infinite series

$$1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + 7 \cdot 8 + \ldots ,$$

where the dot between the two integers of each term indicates multiplication.

Solution: The second factor of each term is evidently twice the number of the term. Thus,

the 11th term is $21 \cdot 22 = 462$.

It is frequently desirable to use letters for the terms of a sequence or a series, and often a subscript is attached to indicate the number of the term counting from the beginning, or from some fixed point. Thus, the most general infinite sequence may be written in the form

$$a_1, a_2, a_3, \ldots , \tag{13-1a}$$

and the most general infinite series as

$$a_1 + a_2 + a_3 + \ldots \quad . \tag{13-1b}$$

The use of dots may turn out to be ambiguous, because the composition of the sequence (or series), if complicated, may not be evident from a few initial terms. To avoid this difficulty, mathematicians frequently write just the general term. This is the k^{th} term, starting from any fixed point. Of course, any letter may be used instead of k ; the letter used is called the dummy variable. Thus in place of Sequence (13-1a), we use set notation and write

$$\{a_k\}_{k=1}^{\infty} \quad . \tag{13-1c}$$

This symbol means that if we replace k in turn by 1, 2, 3, ... we will have the Sequence (13-1a). The upper and lower symbols appearing outside the braces are called upper and lower indexes. If a sequence is finite, say with n terms, then the upper index is n rather than ∞, and the last term is a_n. In the case of an infinite sequence such as (13-1a), there is no last term such as a_∞ because ∞ is not a number. We use ∞ as the upper index in this case simply to indicate that the sequence is infinite.

A similar shorthand notation is used to represent a series. Since a series is an indicated sum we use what is called "summation notation" and represent Series (13-1b) by the symbol

$$\sum_{k=1}^{\infty} a_k \quad .$$

The symbol Σ is the Greek letter "sigma" which corresponds in English to the first letter of the word "sum". The indexes mean the same thing here that they do in the sequence notation. Thus if a series is finite, say with n terms, we write instead

$$\sum_{k=1}^{n} a_k \quad .$$

Example 13-1d: Write out the finite series

$$\sum_{k=5}^{8} 2^k .$$

Solution: To obtain the terms of the series we have only to substitute the sequence of values 5, 6, 7, 8 for k in the general term 2^k. We get

$$2^5 + 2^6 + 2^7 + 2^8 \text{ , or}$$
$$32 + 64 + 128 + 256 \quad .$$

Example 13-1e: The following symbol is merely another notation for the infinite series of Example 13-1c:

$$\sum_{k=1}^{\infty} (2k - 1)(2k) .$$

You have only to write out the first four terms to assure yourself of this. Try it.

We conclude this section by defining the sum of a finite series.

Definition 13-1c: The sum S_n of a finite series is the sum obtained by adding all of its terms.

The subscript n in the symbol S_n for the sum of a finite series indicates that n terms are added. By definition, $S_n = a_1 + a_2 + \ldots + a_n$. The symbol

$$\sum_{k=1}^{n} a_k$$

is also used to denote the sum of an infinite series. It should be emphasized that each of the symbols

$$a_1 + a_2 + \dots + a_n \qquad \sum_{k=1}^{n} a_k$$

has two meanings as follows: each symbol denotes not only the finite series but also its sum. It will always be clear from the context which meaning is intended.

It should be observed that Definition 13-1c does not define the sum of an infinite series. We must postpone the statement of such a definition until the concept of the limit of a sequence has been introduced in Section 13-4.

Example 13-1f: What is the sum of the following series if n is odd? If n is even?

$$\sum_{k=1}^{n} (-1)^k$$

Solution: We obtain the first term of this series by substituting $k=1$ for the dummy variable k in the general term $(-1)^k$. We obtain the second term by substituting $k=2$, etc. Thus the series is

$$-1 + 1 - 1 + 1 - \dots + (-1)^n .$$

If n is odd, the sum is -1; if n is even the sum is 0.

Exercises 13-1

1. Complete each of the following sequences through 7 terms:
   (a) $-1, -4, -7, -10, \dots$
   (b) $3/4, 6/7, 9/10, 12/13, \dots$
   (c) $\sqrt{2}, 2, 2\sqrt{2}, 4, \dots$
   (d) $2 \times 5, 4 \times 10, 8 \times 20, 16 \times 40, \dots$
   (e) $a, \sqrt[3]{a}, \sqrt[5]{a}, \sqrt[7]{a}, \dots$.

2. In the previous problem find the k^{th} term of each sequence.
3. Repeat Problem 1 using the abbreviated notation for a sequence. Thus Part (a) is

$$\{-3k + 2\}_{k=1}^{7} \quad .$$

4. Complete each of the following series through 7 terms:
   (a) $7 - 2 + 7 - 2 + \dots$
   (b) $7 + 0 - 7 + 0 + \dots$
   (c) $a + 2a + 3a + 4a + \dots$
   (d) $1 - 2 + 3 - 4 + \dots$.
5. Find the sum of the first 7 terms of each series in Problem 4.
6. Write each of the following series using the Σ notation:
   (a) $-1 + 1 + 3 + 5 + \dots + 17$
   (b) $2 - 4 + 8 - 16 + \dots - 256$
   (c) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots$
   (d) $(1 + i) + 1 + (1 - i) + (1 - 2i) + \dots$.
7. Write all terms for each of the following:

   (a) $\left\{\frac{k(k + 1)}{2}\right\}_{k=-2}^{4}$

   (b) $\sum_{k=-1}^{2} (-k^3)^3$

   (c) $\sum_{k=-3}^{3} \frac{k(k + 1)}{2}$

   (d) $\{(-2)^{k^2}\}_{k=-2}^{2}$

8. Show by writing out all of the terms that the following symbols all represent the same series:

(a) $$\sum_{k=-3}^{2} \frac{k}{k+4} ,$$

(b) $$\sum_{j=-20}^{-15} \frac{j+17}{j+21} ,$$

(c) $$\sum_{m=14}^{19} \frac{m-17}{m-13} .$$

9. Write the n^{th} term of the series

$$\sum_{k=2}^{\infty} (k^2 - 14k) .$$

10. Admitting that the sequence

$$\left\{\frac{k(k+1)}{2}\right\}_{k=1}^{\infty}$$

has its first two terms odd numbers, its next two even, etc., find the general term of the series

$$1 + 5 - 9 - 13 + 17 + 21 - 25 - 29 + \dots .$$

11. Show that

(a) $$\sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k ,$$

(b) $$\sum_{k=1}^{n} c = nc ,$$

(c) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$,

(d) If $1 < m < n$,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k .$$

12. Is it true that

(a) $\sum_{k=1}^{n} a_k b_k = \left(\sum_{k=1}^{n} a_k\right) \cdot \left(\sum_{k=1}^{n} b_k\right)$?

(b) $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{\sum_{k=1}^{n} a_k}$?

If $n > m > 1$,

(c) $\sum_{k=m}^{n} a_k = \sum_{k=1}^{n-m+1} a_{k+m-1}$?

13-2. Arithmetic Sequences and Series.

Certain sequences and series are of such frequent occurrence that they have been given special names.

Definition 13-2a: An arithmetic sequence is a sequence in which the difference obtained by subtracting any term from its successor is always the same. This difference is called the common difference of the arithmetic sequence and is designated by the letter d .

An arithmetic sequence is also called an arithmetic progression and we say that the terms of the sequence are "in arithmetic progression." The common difference of an arithmetic sequence is obtained by subtracting any term from its successor. Thus if a_1, a_2, a_3, ..., a_{k-1}, a_k, ... are the terms of an arithmetic sequence, then

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_k - a_{k-1} = \ldots = d \ .$$

Following are examples of arithmetic sequences:

(a) 1, 2, 3, ..., 17; $d = 1$.

(b) 17, 12, 7, 2, -3, -8; $d = -5$.

(c) 3, 6, 9, 12, ...; $d = 3$.

The sequence 1, -1, 1, -1, ... is not arithmetic because the differences between successive terms are alternately -2,2; this sequence has no common difference.

From Definition 13-2a it follows that if a_1 is the first term of an arithmetic sequence, then

$$a_2 = a_1 + d \ ,$$

$$a_3 = a_1 + d + d = a_1 + 2d \ ,$$

$$\ldots$$

13-2a $$a_n = a_1 + (n - 1)\, d \ .$$

The last line provides us with an easy formula for finding the n^{th} term of an arithmetic sequence whenever the first term and the common difference are known. We illustrate an application of Formula 13-2a by means of an example.

Example 13-2a: If the 2^{nd} term of an arithmetic sequence is 0 and the 9^{th} term is 14 , what are the 1^{st} and 100^{th} terms?

Solution: Using Formula 13-2a with $a_2 = 0$ and $a_9 = 14$, we see that

$$14 = a_1 + (9 - 1)d ,$$

and

$$0 = a_1 + (2 - 1)d .$$

Whence $\quad d = 2$, $a_1 = -2$, and $a_{100} = 196$.

From the definition of a series given in Section 13-1 (Definition 13-1b) it follows that an arithmetic series is the indicated sum of the terms of an arithmetic sequence. The most important arithmetic series is the one whose terms are the positive integers. The common difference of this series is 1 , and the sum S_n of the first n terms is given by the equation

13-2b $$S_n = 1 + 2 + \dots + (n - 1) + n .$$

Since the sum S_n is not affected by the order of addition, we can reverse the order of the terms on the right side and also write

13-2c $$S_n = n + (n - 1) + \dots + 2 + 1 .$$

Adding Equations 13-2b and 13-2c , we obtain

$$2S_n = (n + 1) + (n + 1) + \dots + (n + 1) + (n + 1) ,$$

and dividing both members by 2 , we have

$$S_n = \frac{n(n + 1)}{2} .$$

Making use of the Σ notation introduced in Section 13-1, the last equation can also be written as

$$\sum_{k=1}^{n} k = \frac{n^2 + n}{2} .$$

The result we have obtained can be stated as a theorem.

Theorem 13-2a:

13-2d $$1 + 2 + 3 + \ldots + n = \sum_{k=1}^{n} k = \frac{n^2 + n}{2} ,$$

where n is a positive integer.

One could almost have guessed this result. Since the difference between successive terms is the same, it is reasonable to suppose that the average term of this series is half the sum of the first and last terms, or half the sum of the second and next-to-last, etc. The sum of n terms, each of which has the average value $\frac{(n + 1)}{2}$, is $\frac{n^2 + n}{2}$ as stated in Equation 13-2d. We shall soon see that this is a general rule for all finite arithmetic series; that is:

Sum = (number of terms) × (average of first and last terms).

Theorem 13-2a may be used to find the sum of any arithmetic series. The following examples are illustrative.

Example 13-2a. Find the sum of the series

$$3 + 7 + 11 + 15 + 19 + 23 .$$

Solution: Subtracting the first term from each of the six terms of the series and compensating by adding an equal quantity, the sum is equal to

$$\begin{aligned} &6(3) + [(3 - 3) + (7 - 3) + (11 - 3) + (15 - 3) \\ &\qquad + (19 - 3) + (23 - 3)] \\ &= 6(3) + (0 + 4 + 8 + 12 + 16 + 20) \\ &= 6(3) + 4(1 + 2 + 3 + 4 + 5) . \end{aligned}$$

Applying Theorem 13-2a to the expression within parenthesis, we find that the sum is

$$6(3) + 4\left(\frac{5^2 + 5}{2}\right) = 78 .$$

Example 13-2b: Find the sum of 500 terms of the arithmetic series

$$(-\frac{3}{4}) + (-1\frac{1}{4}) + (-1\frac{3}{4}) + \dots \; .$$

Solution: To find the sum of this series we need to know the last term. Using Formula 13-2a with $a_1 = -\frac{3}{4}$ and $d = -\frac{1}{2}$, we see that

$$a_{500} = (-\frac{3}{4}) + 499(-\frac{1}{2}) \; .$$

Thus the series can be written in the form

$$[-\frac{3}{4}] + [-\frac{3}{4} + 1(-\frac{1}{2})]$$
$$+ [-\frac{3}{4} + 2(-\frac{1}{2})] + \dots + [-\frac{3}{4} + 499(-\frac{1}{2})] \; .$$

Whence we see that the sum is

$$500(-\frac{3}{4}) + (-\frac{1}{2})\,(1 + 2 + 3 + \dots 499)$$
$$= -375 - \frac{(499)^2 + 499}{4} = -62750 \; .$$

Example 13-2c: If the sum of the first n positive integers is 190, what is n?

Solution: $\frac{n^2 + n}{2} = 190$

$$(n - 19)(n + 20) = 0$$
$$n = 19 \; .$$

The solution $n = -20$ has no meaning in the present context because n is a positive integer.

Although it is possible to find the sum of any arithmetic series by employing the scheme used in the preceding examples, it will be helpful to have formulas that can be applied directly. We can obtain two useful formulas by applying the method of Examples 13-2a and 13-2b in the general case. Thus, consider the arithmetic series of n terms, having first term a_1, common difference d, and n^{th} term $[a_1 + (n - 1)d]$:

$$a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + [a_1 + (n - 1)d] .$$

If we let S_n represent the sum, then

$$S_n = na_1 + [1 + 2 + 3 + \ldots + (n - 1)]d .$$

Applying Theorem 13-2a to the quantity within brackets, we have

$$S_n = na_1 + [\frac{(n - 1)^2 + (n - 1)}{2}] d \text{ , or}$$

13-2e $$S_n = na_1 + \frac{n(n - 1)d}{2} .$$

Combining terms in the right member of 13-2c we get

$$S_n = \frac{2na_1 + n(n - 1)d}{2}$$

$$= \frac{na_1 + na_1 + n(n - 1)d}{2}$$

$$= \frac{na_1 + n[a_1 + (n - 1)d]}{2} .$$

But by Formula 13-2a , $a_n = a_1 + (n - 1)d$; so

13-2f $$S_n = \frac{n}{2}(a_1 + a_n) .$$

Equations 13-2c and 13-2d give us useful formulas for finding the sum of any arithmetic series. Which one we use in a given case depends on what facts we are given. Using these formulas will greatly simplify the work in Examples 13-2a and 13-2b .

Example 13-2d: Find the sum of the series

$$\sum_{k=2}^{16} (k - 15) .$$

Solution: Noting that the lower and upper indexes are respectively 2 and 16, we see that the series has 15 terms. Substituting $k = 2$ and $k = 16$, we find that $a_1 = -13$, $a_{15} = 1$. Using Formula 13-2d we get

$$S_n = \frac{15}{2}(-13 + 1) = -90 .$$

Note. Occasionally the last line is written as

$$\sum_{k=2}^{16} (k - 15) = -90 .$$

When mathematicians use the Σ notation in this way, they have in mind the sum of the series rather than the series itself. Although we introduced the Σ notation as a symbol for a series, the dual usage of this notation should cause no difficulty, because it will usually be clear from context which usage is intended.

Example 13-2e: A body falling from rest in a vacuum falls approximately 16 ft. the first second and 32 ft. farther in each succeeding second. How far will it fall in 11 seconds? In t seconds?

Solution: The series is

$$16 + 48 + 80 + \dots .$$

Using Formula 13-2e with $a_1 = 16$, $d = 32$, and $n = 11$ (or t), we see that for 11 terms

$$S_n = 11 \cdot 16 + \frac{11 \cdot 10 \cdot 32}{2} = 1936 ;$$

and for t terms

$$S_n = 16\,t + \frac{t(t - 1) \cdot 32}{2} = 16\,t^2$$

Exercises 13-2

1. Determine which of the following series are arithmetic. Find d and the next three terms for those that are.
   (a) $4 + 10 + 16 + 22 + \dots$
   (b) $5 + 9 + 12 + 18 + \dots$
   (c) $2.0 + 2.5 + 3.0 + 3.5 + \dots$
   (d) $2 + 3 \cdot 2 + 5 \cdot 2 + 7 \cdot 2 + \dots$
   (e) $-10 - 6 - 2 + 2 + \dots$
2. Find the sum of the series

$$\sum_{k=1}^{16} (k - 17)$$

3. Find the sum of the series

$$\sum_{k=0}^{5} (2k - 4)$$

4. Write the first 5 terms of an arithmetic series in which the second term is m and the third term is p.
5. Write the first five terms of an arithmetic series in which the second term is m and the fourth term is p.
6. If the third term of an arithmetic sequence is -1 and the 16th is $\frac{11}{2}$, what is the first term?
7. If $2 - n$ is the nth term of an arithmetic series, write the first term.
8. Find the indicated term in each of the following series:
   (a) 15th term in $3 + 5 + \dots$
   (b) 11th term in $-2 + 1 + \dots$
   (c) 9th term in $\frac{5}{2} + \frac{13}{2} + \dots$
9. How many integers are there between 35 and 350 which are divisible by 23 ?

10. The arithmetic mean between a and b is $\frac{a + b}{2}$. Find its value if
    (a) $a = 5$, $b = 65$.
    (b) $a = -6$, $b = 2$.
    (c) $a = 3 - \sqrt{3}$, $b = 7 + 5\sqrt{3}$.
    (d) $a = (c + d)^2$, $b = c^2 - d^2$.
11. Take every 5th term from an arithmetic sequence and form a new sequence. Is the new sequence arithmetic?
12. If $3\frac{1}{2}$ and $8\frac{1}{2}$ are the first and eighth terms of an arithmetic sequence, find the six terms that should appear between these two so that all eight terms will be in arithmetic progression. (The six terms you are asked to find are called the six arithmetic means between $3\frac{1}{2}$ and $8\frac{1}{2}$.)
13. Find the sum of the following series by using Theorem 13-2a.
    (a) $1 + 2 + 3 + \ldots + 10$
    (b) $1 + 2 + 3 + \ldots + 999$
    (c) $-3 - 6 - 9 - 12 - 15$
14. On a ship, time is marked by striking one bell at 12:30, two bells at 1:00, three bells at 1:30, etc. up to a maximum of 8 bells. The sequence of bells then begins anew, and it is repeated in each successive interval of four hours throughout the day. How many bells are struck during a day (24 hours)? How many are struck at 10;30 p.m.?
15. Find the sum of the series

$$\sum_{k=1}^{n} (ak + b) .$$

16. Find n if
    $1 + 2 + 3 + \ldots + n = 153$.

17. Find a and b if

$$\sum_{k=0}^{4} (ak + b) = 10 \quad ; \quad \sum_{k=1}^{4} (ak + b) = 14 \ .$$

18. Find the sum of the series

$$\sum_{k=-n}^{m} k \ , \quad m > 0 \ , \ n > 0 \ .$$

Show that the sum is the number of terms multiplied by the average of the first and last terms. (Here k runs through all integers from $-n$ to m inclusive.)

19. The digits of a positive integer having three digits are in arithmetic progression and their sum is 21 . If the digits are reversed, the new number is 396 more than the original number. Find the original number.

20. Find formulas for a_1 and S_n when d, n and a_n are given.

21. Find x if $(3 - x)$, $-x$, $\sqrt{9 - 2x}$ are in arithmetic progression.

22. The sum of three numbers which are in arithmetic progression is -3 and their product is 8 . Find the numbers.

23. Find the sum of all positive integers less than 300 which
    (a) are multiples of 7 ,
    (b) end in 7 .

13-3. Geometric Sequences and Series.

Another very important special sequence is the geometric sequence.

Definition 13-3a: A geometric sequence is a sequence in which the ratio of any term to its predecessor is the same for all terms.

Thus if the first term of a geometric sequence is a_1 and the common ratio is r , then

$$a_2 = a_1r ,$$

$$a_3 = a_1r^2 ,$$

$$\dots$$

13-3a $$a_n = a_1r^{n-1} ,\text{ where } n \text{ is a positive integer.}$$

The last line gives us a formula for the n^{th} term. Geometric sequences are also referred to as geometric progressions, and the terms of the sequence are said to be "in geometric progression".

From the definition of a series given in Section 13-1 (Definition 13-1b) it follows that a geometric series is the indicated sum of the terms of a geometric sequence. For the sequence introduced above we have the geometric series

13-3b $$a_1 + a_1r + a_1r^2 + \dots + a_1r^{n-1} = \sum_{k=0}^{n-1} a_1r^k = a_1 \sum_{k=0}^{n-1} r^k,$$

which is finite and has n terms, or the infinite series

13-3c $$a_1 + a_1r + a_1r^2 + \dots = \sum_{k=0}^{\infty} a_1r^k = a_1 \sum_{k=0}^{\infty} r^k.$$

As with all infinite series, (13-3c) has no last term.

Following are examples of geometric series:

(a) $1 + 2 + 4 + 8 + 16 + \dots ; \quad r = 2 .$

(b) $i - 1 - i + 1 + i - \dots ; \quad r = i .$

(c) $\sqrt{2} + \sqrt{6} + 3\sqrt{2} + 3\sqrt{6} + \dots ; \quad r = \sqrt{3}$

(d) $\frac{10}{3} + 1 + .3 + .09 + .027 ; \quad r = .3 .$

(e) $\pi - \pi^2 + \pi^3 - \pi^4 + \pi^5 - \dots ; \quad r = - \pi .$

(f) $3(10) + 3(10)^0 + 3(10)^{-1} + 3(10)^{-2} + 3(10)^{-3} ; \quad r = \frac{1}{10} .$

The last series is finite so we know that it has a sum. Interestingly enough its sum is 33.333 . This illustrates that any number all of whose digits are identical is really the sum of a geometric series with common ratio $\frac{1}{10}$.

The series

$$1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 + \dots$$

is not geometric because it has no common ratio. The ratios are alternately 1 and -1 .

To obtain a formula for the sum of the finite geometric Series 11-3b recall the following formulas from Chapter 1 for factoring polynomials:

$$1 - r^2 = (1 + r)(1 - r) ,$$
$$1 - r^3 = (1 + r + r^2)(1 - r) .$$

An extension of these formulas to the form $1 - r^n$ suggests that

13-3d $$1 - r^n = (1 + r + r^2 + \dots + r^{n-1})(1 - r) .$$

The equality in the last line can be checked by multiplication.

$$\begin{array}{r}
1 + r + r^2 + \dots + r^{n-1} \\
1 - r \\ \hline
- r - r^2 - r^3 - \dots - r^{n-1} - r^n \\
1 + r + r^2 + r^3 + \dots + r^{n-1} \\ \hline
1 + 0 + 0 + 0 + \dots + 0 - r^n .
\end{array}$$

If $r \neq 1$ we may divide both sides of Equation 13-3d by $(1 - r)$ to get

$$1 + r + r^2 + \dots + r^{n-1} = \frac{1 - r^n}{1 - r} .$$

Multiplying both members of the last equation by a_1 we obtain the sum of Series 13-3b . We have proved the following theorem:

Theorem 13-3a:

13-3e
$$
\begin{aligned}
a_1 + a_1r + a_1r^2 + \dots + a_1r^{n-1} &= \sum_{k=0}^{n-1} a_1r^k \\
&= \frac{a_1(1 - r^n)}{1 - r} \quad \text{if} \quad r \neq 1 \, , \\
&= na_1 \quad \text{if} \quad r = 1 \, ,
\end{aligned}
$$

where $n = 1, 2, \dots$.

Note that if $r = 1$, the series has n terms all equal to a_1 , so that the sum is na_1 . Equation 13-3e can be used as a formula for application problems in which $r \neq 1$. In this connection, however, we usually represent the left member of Equation 13-3e by s_n and employ the shorter form

13-3f
$$s_n = \frac{a_1(r^n - 1)}{r - 1} \; .$$

Another useful formula for the sum of a finite geometric series is

13-3g
$$s_n = \frac{ra_n - a_1}{r - 1} \; .$$

This formula can be easily obtained by making use of Formula 13-3a. Since

$$a_n = a_1r^{n-1} \, ,$$

$$ra_n = a_1r^n \; .$$

Rewriting 13-3f in the form

$$s_n = \frac{a_1r^n - a_1}{r - 1} \, ,$$

and substituting $ra_n = a_1r^n$ we have 13-3g .

Example 13-3a: If the 4^{th} term of a geometric series is 6 and the 9^{th} term is 1458, find the 1^{st} term, the 10^{th} term, and the sum of the first ten terms.

Solution: Use Formula 13-3a twice: first with $n = 4$, $a_4 = 6$; then with $n = 9$, $a_9 = 1458$. We get two equations:

$$6 = a_1 r^3$$

$$1458 = a_1 r^8 \quad .$$

Solving, we obtain $r = 3$, $a_1 = \frac{2}{9}$; from which $a_{10} = 4374$. Using Formula 13-3f with $n = 10$, we get

$$s_n = \frac{2}{9}\left(\frac{3^{10} - 1}{3 - 1}\right) = 3^8 - \frac{1}{9} = 6560\,\frac{8}{9} \ .$$

Example 13-3b: If a finite geometric series has the last term 1296, ratio 6, and a sum of 1555, find the 1^{st} term.

Solution: Using Formula 13-3g with $a_n = 1296$, $r = 6$, and $s_n = 1555$, we have

$$1555 = \frac{6(1296) - a_1}{6 - 1} \ .$$

Hence $a_1 = 1$.

Exercises 13-3

1. Write the next three terms in each of the following geometric sequences:
   (a) $-2, -10, \ldots,$
   (b) $-\frac{9}{8}, \frac{3}{4}, \ldots,$
   (c) $7, 1, \ldots$.
2. If $a + b + c$ is a geometric series, express b in terms of a and c.

3. Find the sum of the following series:
   (a) $1 + 2 + 2^2 + \dots + 2^9$,
   (b) $1 - 3 + 3^2 - 3^3 + 3^4$,
   (c) $1 - 1 + 1 - 1 + \dots + 1 - 1$ (100 terms) .
4. Find the sum of the series

$$\sum_{k=5}^{99} r^k \ .$$

5. Find n if $\sum_{k=0}^{n} 2^k = 63$.
6. Find n if $3 + 3^2 + 3^3 + \dots + 3^n = 120$.
7. Can two different geometric series have the same sum, the same first term, and the same number of terms? (Try

$$1 + r + r^2 = 7 \ .)$$

8. Find the sum of the series

$$\sum_{k=n}^{n+r} r^k \text{ , both when } r = 1 \text{ and when } r \neq 1 \ .$$

9. Find the sum of the series

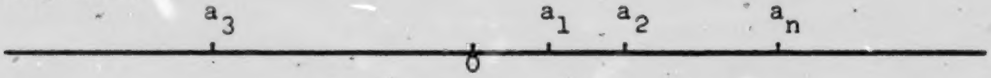
$$\sum_{k=0}^{n} 2^{2k+1} \ .$$

10. Find the numbers x to make the following series geometric:
    (a) $-\frac{3}{2} + x - \frac{8}{27} + \dots$,
    (b) $\sqrt{2 - x} + \sqrt{20 - x} + \sqrt{18 - 9x} + \dots$.
11. How many terms are there in the geometric series
    $32 + 16 + 8 + \dots + \frac{1}{256}$?

12. Find, if possible, the 1^{st} and 2^{nd} terms of a geometric series with 3^{rd} term = -4, 5^{th} term = -1, 8^{th} term = $-\frac{1}{8}$.
13. Find all sets of 3 integers in geometric progression whose product is -216 and the sum of whose squares is 189.
14. If M is the foot of a perpendicular drawn from a point P of a semicircle to the diameter $\overline{AB}$, show that lengths $\overline{AM}$, $\overline{MP}$, $\overline{MB}$ are in geometric progression.
15. The terms of a finite geometric series between the first and the last are called *geometric means* between the first and last. If the series has only three terms, the middle term is called *the geometric mean* between the other two. Insert
    - (a) 3 geometric means between 1 and 256,
    - (b) 2 geometric means between $\sqrt{5}$ and 5,
    - (c) the geometric mean between a^8 and $16b^4$,
    - (d) the geometric mean between a and b.

13-4. Limit of a Sequence.

Recall again the definition of a sequence of numbers stated in Section 13-1 (Definition 13-1a). We will find it convenient to plot the numbers $a_1, a_2, a_3, \ldots a_n, \ldots$ on a number line. To avoid confusion we will label the points associated with the numbers of the sequence by the symbols which represent them in the sequence.

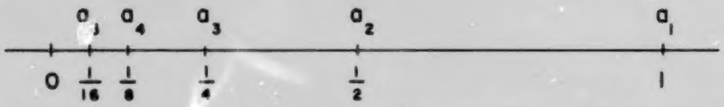


In this way we can establish a correspondence between the terms of a sequence of numbers and a set of points on the number line.

To study the behavior of a sequence of numbers and the points corresponding to them let us look briefly at several examples.

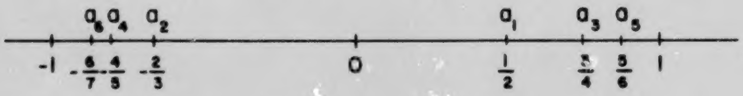
Example 13-4a:

(a) $1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8} , \frac{1}{16} , \ldots$


a_5
a_4
a_3
a_2
a_1
0
$\frac{1}{16}$
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
1

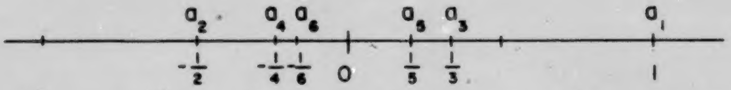
Points corresponding to successive terms of the sequence get closer and closer to the point 0 as n becomes large; that is, a_n approaches zero as n becomes large.

(b) $\frac{1}{2} , -\frac{2}{3} , \frac{3}{4} , -\frac{4}{5} , \frac{5}{6} , -\frac{6}{7} , \ldots$


a_6
a_4
a_2
a_1
a_3
a_5
-1
$-\frac{6}{7}$
$-\frac{4}{5}$
$-\frac{2}{3}$
0
$\frac{1}{2}$
$\frac{3}{4}$
$\frac{5}{6}$
1

If n is odd, the points corresponding to a_n get closer and closer to 1 as n becomes large; if n is even, the points corresponding to a_n approach -1 . Hence a_n alternately approaches 1 and -1 .

(c) $1 , -\frac{1}{2} , \frac{1}{3} , -\frac{1}{4} , \frac{1}{5} , -\frac{1}{6} , \ldots$


a_2
a_4
a_6
a_5
a_3
a_1
$-\frac{1}{2}$
$-\frac{1}{4}$
$-\frac{1}{6}$
0
$\frac{1}{5}$
$\frac{1}{3}$
1

Points corresponding to a_n are alternately to the right and left of 0 ; however, as n becomes large successive points get closer and closer to 0 . Hence, a_n approaches zero as n becomes large.

(d) $1 , -1 , 1 , -1 , 1 , -1 , \ldots$

In this case a_n is alternately equal to 1 and -1 .

(e) $1 , \sqrt{2} , \sqrt{3} , 2 , \sqrt{5} , \sqrt{6} , \ldots$

In this case it is easy to see that as n becomes large so does a_n .

The foregoing examples show clearly that there are two kinds of sequences which differ according to the way in which a_n behaves as n becomes large. In both (a) and (c) we see that as n becomes large a_n approaches some fixed number A, and we say that a_n approaches a limit A as n becomes large. Such sequences are said to be convergent. On the other hand, in (b), (d), and (e), there is no fixed number that a_n approaches. Sequences of this kind are said to be divergent.

The notion of limit may be familiar to you. In geometry, for example, you learned that the area of a regular polygon inscribed in a circle approaches the area of the circle as the number of sides increases. In Section 6-6 of this text it was shown that the distance between a point on a branch of a hyperbola and an asymptote approaches zero as the point moves out indefinitely far on the curve.

To make the notion of the limit of a sequence precise, we state the following definition.

Definition 13-4a: The sequence $a_1, a_2, a_3, \ldots$ has a limit A if a_n becomes and remains arbitrarily close to A as n gets larger and larger. A sequence that has a limit is said to be convergent.

Under the conditions of the definition we also say that "the limit of a_n as n becomes infinite is A," and we write the statement which appears in quotation marks with the symbol

$$\lim_{n \to \infty} a_n = A .$$

The following examples illustrate the definition. The limit in each case is given. Write enough additional terms in each example to satisfy yourself that the given sequence has the indicated limit.

Example 13-4b:

$$1 , \frac{1}{2} , \frac{1}{3} , \frac{1}{4} , \ldots , \frac{1}{n} , \ldots ; A = 0 .$$

Although it may seem obvious that the limit of the given sequence should be zero, this example is not trivial and will be useful later. Symbolically we ordinarily write

$$\lim_{n \to \infty} \left(\frac{1}{n}\right) = 0 .$$

Example 13-4c:

$$1 , \frac{1}{2} , 1 , \frac{3}{4} , 1 , \frac{7}{8} , 1 , \ldots ; A = 1 .$$

Here $a_n = 1$ when n is odd, and $a_n = 1 - 2^{-\frac{n}{2}}$ when n is even. It is not ruled out by the definition that a_n may be equal to its limit for some values of n, or even for infinitely many values of n.

Example 13-4d:

$$(2 + \tfrac{1}{2}) , (2 - \tfrac{1}{2}) , (2 + \tfrac{1}{3}) , (2 - \tfrac{1}{3}) , (2 + \tfrac{1}{4}) ,$$
$$(2 - \tfrac{1}{4}) , \ldots ; \quad A = 2 .$$

Note that a_n is alternately larger and smaller than A. Sequence (c) in Example 13-4a behaves similarly.

Note: A fact which deserves mention at this point is that a sequence cannot have two different limits, because it is not possible for a_n to be arbitrarily close to each of two different numbers for all n sufficiently large. What is meant here is illustrated by sequence (b) in Example 13-4a. This sequence, as already stated, is not convergent; it is divergent.

In the preceding examples the limit for each sequence was given and it was relatively easy to see that the indicated limit was indeed the limit of the given sequence. On the other hand, determining whether a given sequence has a limit and finding its value when there is one calls for some specialized knowledge of the properties of limits. In advanced courses in mathematics these properties are usually stated as theorems. Before stating such theorems we first observe that the sequences

$$a_1 , a_2 , a_3 , \ldots a_n , \ldots ,$$

$$b_1 , b_2 , b_3 , \ldots b_n , \ldots ,$$

can be used to form many new sequences; for example,

$$a_1 + b_1 , a_2 + b_2 , a_3 + b_3 , \ldots , a_n + b_n , \ldots ,$$

$$a_1b_1 , a_2b_2 , a_3b_3 , \ldots , a_nb_n \ldots ,$$

$$\frac{a_1}{b_1} , \frac{a_2}{b_2} , \frac{a_3}{b_3} , \ldots , \frac{a_n}{b_n} , \ldots \text{ (if each } b_n \neq 0) ,$$

$$|a_1| , |a_2| , |a_3| , \ldots , |a_n| , \ldots ,$$

etc.

We conclude this section by stating without proof the following theorems involving limits. The student will find these useful in finding the limit of a sequence.

Theorem 13-4a: The constant sequence $c , c , c , \ldots$ has c as its limit; that is

$$\lim_{n \to \infty} c = c .$$

Theorem 13-4b: If $\lim_{n \to \infty} a_n = A$, and $\lim_{n \to \infty} b_n = B$, then

(1) $\lim_{n \to \infty} (ca_n) = c \lim_{n \to \infty} a_n = cA;$

(2) $\lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n = A \pm B;$

(3) $\lim_{n \to \infty} (a_n b_n) = (\lim_{n \to \infty} a_n)(\lim_{n \to \infty} b_n) = AB;$

(4) $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{A}{B}, \quad b_n \neq 0, \ B \neq 0.$

Example 13-4e: Find the limit of the sequence for which

$$a_n = \frac{1}{n^2} .$$

Solution: Since

$$\frac{1}{n^2} = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) ,$$

the given sequence is the product of two sequences having n^{th} terms $\frac{1}{n}$ and $\frac{1}{n}$. Thus, by Theorem 13-4b(3) and Example 13-4b,

$$\lim_{n \to \infty} \frac{1}{n^2} = \left(\lim_{n \to \infty} \frac{1}{n}\right)\left(\lim_{n \to \infty} \frac{1}{n}\right) = 0 \cdot 0 = 0 .$$

Example 13-4f: Find the limit of the sequence for which

$$a_n = \frac{n}{n+1} .$$

Solution: Dividing the numerator and denominator by n we see that

$$\frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$

Thus, the given sequence is the quotient of two sequences whose n^{th} terms are 1 and $1 + \frac{1}{n}$ respectively.

By Theorem 13-4b(d) we see that

$$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = \frac{\lim\limits_{n \to \infty} 1}{\lim\limits_{n \to \infty} (1 + \frac{1}{n})} .$$

But $\lim\limits_{n \to \infty} (1 + \frac{1}{n}) = \lim\limits_{n \to \infty} 1 + \lim\limits_{n \to \infty} \frac{1}{n} = 1 + 0 = 1$

by Theorem 13-4b(b), 13-4a, and Example 13-4b.

Therefore, $$\lim_{n \to \infty} \frac{n}{n+1} = 1 .$$

Example 13-4g: Find $\lim\limits_{n \to \infty} \frac{1 + 2n + 5n^3}{7n^3}$.

Solution: Dividing the numerator and denominator by n^3 , we note that

$$\frac{1 + 2n + 5n^3}{7n^3} = \frac{\frac{1}{n^3} + \frac{2}{n^2} + 5}{7} .$$

Hence, $$\lim_{n \to \infty} \frac{1 + 2n + 5n^3}{7n^3} = \frac{0 + 0 + 5}{7} = \frac{5}{7} .$$

Example 13-4h: Find the limit of the sequence

$$\frac{5}{3}, \frac{4}{3}, \frac{7}{4}, \frac{6}{4}, \frac{9}{5}, \frac{8}{5}, \frac{11}{6}, \ldots .$$

Solution: If n is an odd integer, say $2k + 1$, then

$$a_n = a_{2k+1} = \frac{2k+5}{k+3} . \quad \text{Also } k \to \infty \text{ as } n \to \infty.$$

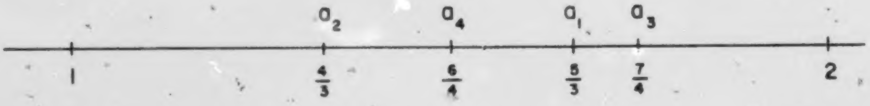
$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} \frac{2k+5}{k+3} = \lim_{k\to\infty} \frac{2+\frac{5}{k}}{1+\frac{3}{k}} = 2 .$$

If n is an even integer, say $2k$, then

$$a_n = a_{2k} = \frac{2k+2}{k+2} . \quad \text{Again } k \to \infty \text{ as } n \to \infty.$$

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} \frac{2k+2}{k+2} = \lim_{k\to\infty} \frac{2+\frac{2}{k}}{1+\frac{2}{k}} = 2 .$$

The novel feature of this example is that as n increases a_n alternately gets closer to and farther from its limit. But the a_n does, none the less, "become and remain arbitrarily close" to 2. An appreciation of what happens when n becomes infinite may be visualized by plotting successive values of a_n on a number line.



Exercises 13-4

1. Evaluate the limit for each sequence that is convergent.

(a) $\left\{\frac{k}{2k+1}\right\}_{k=1}^{\infty}$

(b) $\left\{\frac{k+1}{k}\right\}_{k=1}^{\infty}$

(c) $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \ldots$

*(d) $3, 2, \frac{9}{5}, \frac{12}{7}, \ldots$

2. Make use of Definition 13-4a to decide which of the following sequences converge. Make a guess as to the limit for those that converge.

   (a) $0, 1, 0, 2, 0, 3, 0, 4, \ldots$

   (b) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

   (c) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

   (d) $-1, 2, -3, 4, -5, \ldots$

   (e) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

   (f) $(1 + \frac{1}{2}), (1 + \frac{1}{2} + \frac{1}{4}), (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}), \ldots$

   (g) $1, \frac{1}{2}, -1, \frac{1}{3}, 1, \frac{1}{4}, -1, \frac{1}{5}, \ldots$

   (h) $0.6, 0.66, 0.666, 0.6666, \ldots$

   (i) $2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \ldots$

   (j) $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$

   (k) $1^2, 2^2, 3^2, 4^2, \ldots$

3. Find the following limits:

   (a) $\lim\limits_{n \to \infty} \frac{3n - 1}{2n + 4}$,

   (b) $\lim\limits_{n \to \infty} \frac{n^2 - 2n + 1}{n^2 + n - 1}$,

   (c) $\lim\limits_{n \to \infty} \frac{3n^3 - n}{5n^3 + 17}$,

   (d) $\lim\limits_{n \to \infty} \frac{3n}{n^2 + 1}$,

   (e) $\lim\limits_{n \to \infty} \frac{8}{n^3} \left[\frac{(n - 1)n(2n - 1)}{6}\right]$.

4. Show that

   (a) $\lim\limits_{n \to \infty} \frac{1}{n^2} = 0$,

   (b) $\lim\limits_{n \to \infty} (\frac{1}{n} - \frac{2}{n^2}) = 0$.

5. Prove that

$$\lim_{n \to \infty} \frac{2n^2 - 3n}{4n + 1}$$

   does not exist.

6. Prove that

$$\lim_{n \to \infty} \frac{2n^3}{5n - 1}$$

   does not exist.

7. Prove that

$$\lim_{n \to \infty} \frac{an^2 + bn + c}{dn^2 + en + f} = \frac{a}{d} , \qquad \text{if } d \neq 0 .$$

   Compare with results in 3(a) 3(b) and 3(c) .

8. If $d = 0$ in Exercise 6, can the limit still exist for certain values of the constants? Compare with results in Exercises 5 and 6.

*9. Prove that for any positive integer r

$$\lim_{n \to \infty} \frac{a_0 n^r + a_1 n^{r-1} + \ldots + a_r}{b_0 n^r + b_1 n^{r-1} + \ldots + b_r} = \frac{a_0}{b_0} ,$$

   if $b_0 \neq 0$.

10. Admitting that $\lim\limits_{n \to \infty} r^n = 0$ $(|r| < 1)$, find

$$\lim_{n \to \infty} \frac{1 - r^n}{1 - r} .$$

11. Find

$$\lim_{n \to \infty} \frac{(n^2 - 3n + 5)(3n^3 - 1)}{n(n^4 - 17n + 11)} .$$

Do not expand the products.

12. Find the following limits if convergent.

(a) $\lim\limits_{n \to \infty} 1$

(b) $\lim\limits_{n \to \infty} 0$

(c) $\lim\limits_{n \to \infty} 7$

13-5. Sum of an Infinite Series.

In this section we will make use of the concept of the limit of a sequence developed in the last section to formulate a definition for the sum of an infinite series.

Recall that the definition given in Section 13-1 for the sum of a series applies only to finite series. Even so you may have an intuitive idea of what is wanted in the infinite case. For example, if you meet the number $.\overline{3}$, you may feel quite sure that the number $\frac{1}{3}$ is intended. The infinite decimal is, of course, equivalent to the series.

13-5a $$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots ,$$

and presumably the sum of an infinite series should be defined in such a way that this series will have the sum $\frac{1}{3}$. We emphasize that we have the right to make definitions as we like, if only we agree to stick to the terminology we adopt. However, we also want to keep things reasonable and consistent. For example, if we have a finite series

13-5b $$s_n = a_1 + a_2 + \dots + a_n ,$$

then the sum has already been defined. But s_n in 13-5b can also be regarded as an infinite series all of whose terms are zero after a certain point. A definition of the sum of an infinite series must certainly not conflict with our previous definition in this special case.

Suppose that we are confronted with a special infinite series, say

13-5c $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$$
$$= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} ,$$

and are asked to guess what its sum is. We might proceed as follows. Denoting the sum of the first n terms by s_n, we observe that

$$s_1 = \frac{1}{2} ,$$
$$s_2 = \frac{2}{3} ,$$
$$s_3 = \frac{3}{4} ,$$
$$s_4 = \frac{4}{5} ,$$
$$\dots ,$$
$$s_n = \frac{n}{n+1} ,$$
$$\dots .$$

The numbers s_n listed above are called partial sums; they are partial sums of Series 13-5c. If you look carefully at them you may have the feeling that as n increases s_n approaches some number A which ought to be called the sum of infinite

Series 13-5c. Thus we are faced with the task of determining such a number A if it exists. Recall that in Section 13-4 you learned how to find the limit of a sequence. Let us make use of this fact. Clearly the successive partial sums of Series 13-5c form a sequence; it is called the sequence of its partial sums, and we write it down as follows:

13-5d $$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots .$$

By Definition 13-4a this sequence has a limit and by Example 13-4f we know that its limit is 1 . We propose that this limiting value be the sum of Series 13-5c.

Consider the following infinite series:

13-5e $$1 - 1 + 1 - 1 + \ldots = \sum_{k=0}^{\infty} (-1)^k .$$

Its sequence of partial sums is

$$1, 0, 1, 0, \ldots .$$

Clearly this sequence has no limit, and it does not seem reasonable to call any number the sum of the infinite series.

The last two examples show us that there are two kinds of series which differ according to the way in which their partial sums behave. The first type is said to be convergent, the second divergent.

We are now in a position to define the sum of an infinite series.

Definition 13-5a: The sum of an infinite series is the limit of the sequence of its partial sums if this limit exists. A series which has a sum is called convergent. If no limit exists, the sum of the infinite series is not defined, and the series is said to be divergent.

Let us review the argument used in the introductory examples which precede the definition.

(1) In each case we sought a number which we might call the sum of an infinite series such as

$$13\text{-}5f \quad a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k .$$

(2) We used Definition 13-1c for the sum of a finite series to generate a sequence of partial sums of the given infinite series. That is, we used

$$13\text{-}5g \quad s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

to obtain the sequence of partial sums

$$13\text{-}5h \quad s_1 , s_2 , s_3 , \ldots = \{s_n\}_{n=1}^{\infty} .$$

(3) Finally, we examined the sequence of partial sums for convergence. By the definition which we have just stated (Definition 13-5a) we know that if the sequence of partial sums of a given infinite series has a limit, this limit is the sum of the infinite series. If the sequence of partial sums has no limit then the series does not have a sum.

At first thought one might conclude that we are now equipped with a general method for investigating any infinite series and obtaining its sum, if it has one. This is true, but the method outlined above is not generally useful because of technical difficulties. Except for some easy special cases the method is hard to apply. The difficulty lies in determining an expression for s_n whose limit can be calculated.

Thus, mathematicians rely on far more powerful techniques. Unfortunately, the background on which these depend has not been developed in this text so we shall not introduce them. Thus, in this text, the work of finding sums of infinite series is limited to series that can be handled with the methods presented.

Example 13-5a: Find the partial sums of the series

$$1 + 2 + 3 + \dots = \sum_{k=1}^{\infty} k \, .$$

Solution: Using Definition 13-1c we obtain the sequence

$$1 , 3 , 6 , 10 , \dots = \{\frac{k^2 + k}{2}\}_{k=1}^{\infty} \, .$$

Example 13-5b: Find the sum (if there is one) of the series

$$1 + (\frac{1}{2} - 1) + (\frac{1}{3} - \frac{1}{2}) + \dots = 1 + \sum_{k=1}^{\infty} (\frac{1}{k+1} - \frac{1}{k}) \, .$$

Solution: $s_1 = 1$, $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{3}$, ... , $s_n = \frac{1}{n}$.

By Example 13-4b the desired limit is 0 . Hence the series converges and has the sum 0 .

Example 13-5c: Find the sum (if there is one) of the series

$$2 + \frac{1}{2} + (1 + \frac{1}{3} - \frac{1}{2}) + \dots = 2 + \sum_{k=1}^{\infty} (1 + \frac{1}{k+1} - \frac{1}{k}) \, .$$

Solution: The general term of this series is a unit more than the general term of the previous series.

Hence $s_n = n + \frac{1}{n}$.

But $\lim_{n \to \infty} (n + \frac{1}{n})$

does not exist; that is, the series diverges and has no sum.

Example 13-5d: The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ can be proved to be divergent.

Solution: The partial sums are

$$s_1 = 1 = 2(\tfrac{1}{2}) \ ,$$

$$s_2 = 1 + \tfrac{1}{2} = 3(\tfrac{1}{2}),$$

$$s_4 = (1 + \tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) > 4(\tfrac{1}{2}) \ ,$$

$$s_8 = (1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \dots + \tfrac{1}{8}) > 5(\tfrac{1}{2}) \ ,$$

$$s_{16} = (1 + \tfrac{1}{2} + \dots + \tfrac{1}{8}) + (\tfrac{1}{9} + \dots + \tfrac{1}{16}) > 6(\tfrac{1}{2}) \ ,$$

$$\dots \ .$$

Starting at some point $\frac{1}{n}$, where n is some power of 2, $(n = 2^m)$, examine the next block of 2^m terms,

$$\frac{1}{2^m + 1} + \frac{1}{2^m + 2} + \dots + \frac{1}{2^m + 2^m} \ .$$

Each of these is certainly greater than or equal to $\frac{1}{2^m + 2^m}$. Hence, this block of terms

$$\geq \frac{2^m}{2^m + 2^m} = \frac{1}{2} \ .$$

Since we can find an infinite number of such blocks, the sum has no limit; that is, the series is divergent.

Example 13-5e: On the other hand the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent, and comparison with Problem 19 in the exercises will indicate that this is the series for $\log_e 2$.

We conclude this section by considering a problem which arises as an interesting sidelight to the concepts that have been developed. Suppose that we are given any sequence of numbers. Can we construct a series such that the given sequence is a sequence of partial sums for the series? The answer is yes, and we shall make the matter clear with an example.

Example 13-5f: Construct a series whose partial sums correspond to the sequence given in Example 13-4h:

$$\frac{5}{3} , \frac{4}{3} , \frac{7}{4} , \frac{6}{4} , \frac{9}{5} , \frac{8}{5} , \frac{11}{6} , \ldots \ .$$

Solution: Let $s_1 = \frac{5}{3}$, $s_2 = \frac{4}{3}$, $s_3 = \frac{7}{4}$, ...

Since $s_1 = a_1$,

$$s_2 = a_1 + a_2 = s_1 + a_2 ,$$

$$\ldots ,$$

$$s_n = a_1 + a_2 + \ldots + a_{n-1} + a_n = s_{n-1} + a_n ,$$

we see that

$$a_1 = s_1 ,$$

$$a_2 = s_2 - s_1 ,$$

$$\ldots ,$$

$$a_n = s_n - s_{n-1} .$$

Thus the required series can be obtained from

$$s_1 + (s_2 - s_1) + \ldots + (s_n - s_{n-1}) + \ldots \ .$$

Making the proper substitutions for s_1 , s_2 , etc., we have the series

$$\frac{5}{3} - \frac{1}{3} + \frac{5}{12} - \frac{4}{16} + \frac{6}{20} - \frac{1}{5} + \frac{7}{30} - \ldots$$

$$= \frac{5}{3} - \frac{1}{3} + \frac{5}{3.4} - \frac{1}{4} + \frac{6}{4.5} - \frac{1}{5} + \frac{7}{5.6} - \ldots \ .$$

The last line suggests how to continue the series.

Infinite series are one of the most important tools of advanced mathematics. For example, it can be shown that

$$e^{x} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots, \quad n! = 1 \cdot 2 \cdot 3 \dots n ,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots , \quad x \text{ in radians} ,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots , \quad x \text{ in radians} .$$

Furthermore, these series converge for every value of x, and the sum of each series for any x is the value of the function on the left for that value of x. Infinite series are also important in the calculation of tables of logarithms and tables of trigonometric functions.

Example 13-5g: Find the value of e correct to four decimal places.

Solution: As indicated above,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots .$$

The partial sum s_n is a good approximation to the sum of an infinite series if n is large. We find

$$\begin{array}{rl}
1 = & 1.000000 \\
1 = & 1.000000 \\
1/2! = & .500000 \\
1/3! = & .166667 \\
1/4! = & .041667 \\
1/5! = & .008333 \\
1/6! = & .001389 \\
1/7! = & .000198 \\
1/8! = & .000025 \\
\hline
S_8 \approx & 2.718279
\end{array}$$

If S_8 is rounded off to four decimal places, we obtain 2.7183, which is the value of e correct to four decimal places.

Exercises 13-5

1. Find the partial sum for the first n terms of the series $2 + 7 + 12 + 17 + \dots$.
2. Find the partial sum for the first n terms of the series $\frac{7}{100} + \frac{7}{1000} + \frac{7}{10,000} + \dots$.
3. Find the partial sum for the first n terms of the series $15 + 12 + 9 + 6 + \dots$.
4. Find the partial sum for the first n terms of the series $\frac{1}{2} + 3 + \frac{9}{2} + \frac{27}{4} + \dots$.
5. Find a series whose partial sums are $2, 6, 12, 20, 30, \dots$.
6. Find a series whose partial sums are $2, 6, 14, 30, 62, \dots$.
7. Find a series whose partial sums are $2, -2, 6, -10, 22, \dots$.
8. Find a series whose partial sums are $3, 8, 15, 24, \dots$.
9. Find a series whose partial sums are $2, 8, 20, 40, 70, \dots$.
10. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{3}{(3n + 2)(3n - 1)} .$$

11. Find a series whose partial sums S_k are given by the formula

$$S_k = \frac{k}{2(3k + 2)} .$$

12. Find the sum of the series

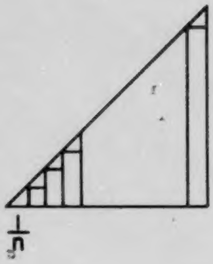
$$\sum_{k=1}^{\infty} \left(\frac{2}{k^2 + 2k + 1} - \frac{2}{k^2}\right) .$$

13. Show that the series $\sum_{k=1}^{\infty} \left(\frac{k^2 + 2k + 1}{k + 3} - \frac{k^2}{k + 2}\right)$ diverges.
14. Determine whether the series $1 - 2 + 3 - 4 + 5 - 6 + \dots$ diverges or converges.

15. The area under $y = x$ from $x = 0$ to $x = 1$, approximated by n rectangles is the sum

$$\frac{1}{n}\left(\frac{0}{n} + \frac{1}{n} + \dots + \frac{n-1}{n}\right) = \frac{1}{n^2}\sum_{k=0}^{n-1} k .$$

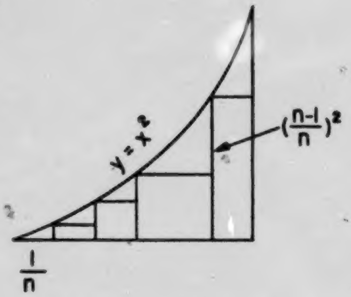
Find the $\lim\limits_{n \to \infty} \frac{1}{n^2} \sum\limits_{k=0}^{n-1} k$.



16. The area under $y = x^2$ from $x = 0$ to $x = 1$, approximated by n rectangles is the sum

$$\frac{1}{n}\left[\frac{0}{n^2} + \frac{1}{n^2} + \frac{4}{n^2} + \dots \frac{(n-1)^2}{n^2}\right]$$

$$= \frac{1}{n^3}\sum_{k=0}^{n-1} k^2 .$$



If $\sum\limits_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, find the $\lim\limits_{n \to \infty} \frac{1}{n^3} \sum\limits_{k=0}^{n-1} k^2$.

17. Find the limit of the sum (if there is one) of

$$\frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \frac{2}{7 \cdot 9} + \dots .$$

Hint: $\frac{2}{(2k + 1)(2k + 3)} = \frac{1}{2k + 1} - \frac{1}{2k + 3}$.

18. With the series given for e^x, find the approximation for $\sqrt{e}$.

19. An approximation for the natural logarithm is given by the following series for $-1 < x \leq 1$.

$$\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots .$$

Use this series to estimate $\log_e 1 \cdot 1$ to 4-place decimals. Compare this result with the appropriate area under the curve $y = \frac{1}{x}$ in Chapter 9.

13-6. The Infinite Geometric Series.

No infinite arithmetic series converges unless all of its terms are zero; hence, the convergence of infinite arithmetic series will not be considered further. On the other hand, we have already seen that certain infinite geometric series may converge. For example, the infinite decimal $.\overline{3}$ mentioned at the beginning of Section 13-5 has the value $\frac{1}{3}$, and this is equivalent to saying that a certain geometric series converges and has the sum $\frac{1}{3}$.

By Theorem 13-3a the n^{th} partial sum s_n of the infinite geometric series

$$a_1 + a_1r + a_1r^2 + \dots = \sum_{k=0}^{\infty} a_1r^k \qquad \text{13-6a}$$

is

$$s_n = a_1\left(\frac{1 - r^n}{1 - r}\right), \qquad \text{if } r \neq 1 .$$

Can this partial sum have a limit as n becomes infinite? It depends on r^n. If $r = 2$, then $r^2 = 4$, $r^3 = 8$, etc., and r^n increases rapidly as n increases. No limit exists for r^n, nor for s_n. If $r = -2$, then $r^2 = 4$, $r^3 = -8$, etc., and again no limit exists. On the other hand, if $r = \pm 1/2$, then $r^2 = 1/4$, $r^3 = \pm 1/8$, etc., and r^n approaches zero. That is, s_n will have the limit $\frac{a_1}{1 - r}$. The result is evidently going to depend on the absolute value of r. The above argument shows that if $|r| > 1$ then Series 13-6a diverges; if $|r| < 1$ then Series 13-6a converges and has the sum $\frac{a_1}{1 - r}$. Finally, if $r = \pm 1$ the series reduces to

$$a_1 + a_1 + a_1 + \dots ,$$

or

$$a_1 - a_1 + a_1 - \dots .$$

These certainly diverge unless $a_1 = 0$. We summarize our results by stating a theorem.

Theorem 13-6a: The infinite geometric Series 13-6a converges and has the sum $\frac{a_1}{1 - r}$ if $|r| < 1$. It diverges if $|r| \geq 1$ (unless $a_1 = 0$, when it converges).

Example 13-6a: Find the sum of the series

$$1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \dots = \sum_{k=0}^{\infty} (- \tfrac{1}{4})^k .$$

Solution: $a_1 = 1$, $r = -\frac{1}{4}$, $|r| < 1$. The series converges and has the sum $\frac{1}{1 + \frac{1}{4}} = \frac{4}{5}$.

Example 13-6b: Find the value of the repeating decimal $.\overline{142857}$.

Solution: This is equivalent to the geometric series

$$\frac{b}{10^6} + \frac{b}{10^{12}} + \frac{b}{10^{18}} + \dots , \qquad b = 142857 .$$

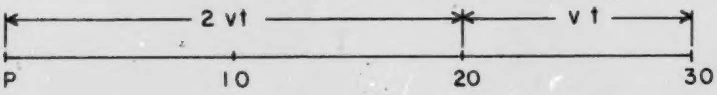
$$a_1 = b(10^{-6}) , \ r = 10^{-6} < 1 .$$

The series converges and has the sum

$$\frac{b(10^{-6})}{1 - 10^{-6}} = \frac{b}{10^6 - 1} = \frac{142857}{999999} = \frac{1}{7} .$$

Example 13-6c: A train is approaching a point P 30 miles away, at 30 miles per hour. A fly with twice the speed leaves P , touches the train, returns to P and repeats the process until the train reaches P . How far does the fly travel?

Solution: To simplify matters let v represent the velocity of the train, even though we know that $v = 30$, and let $2v$ represent the velocity of the fly. Suppose the first meeting is t_1 hours after the start. Thus after t_1 hours the fly will be $2vt_1$ miles from P and the train will be $30-vt_1$ miles from P.



Hence $30-vt_1 = 2vt_1$, $vt_1 = 10$; and since $v = 30$, $t_1 = \frac{1}{3}$. So the fly has traveled 20 miles and the train 10 miles when they meet for the first time. Therefore the fly's first round trip is $2(2v)(\frac{1}{3}) = \frac{4v}{3}$ miles. The train is now 10 miles from P, and we repeat the computation. Let t_2 be the time required for the fly to go from P to the train the second time. Thus $10-vt_2 = 2vt_2$, $vt_2 = \frac{10}{3}$; and since $v = 30$, $t_2 = \frac{1}{3^2}$. The fly's second round trip is $2(2v)(\frac{1}{3^2}) = \frac{4v}{3^2}$, etc. The answer, in series form, is

$$\frac{4v}{3} + \frac{4v}{3^2} + \frac{4v}{3^3} + \dots ,$$

where $a_1 = \frac{4v}{3}$, $r = \frac{1}{3} < 1$. By Theorem 13-6a the sum is

$$\left(\frac{4v}{3}\right)\left(\frac{1}{1-\frac{1}{3}}\right) = 2v = 60 .$$

The fly travels 60 miles. This result can be checked directly without using series. We have only to note that the train needs one hour to get to P, and if the fly wastes no time it can do 60 miles in that time. We have deliberately done this example the hard way to illustrate Theorem 13-6a for a case in which we know the answer in advance.

Exercises 13-6

1. Find the sum of the series

   (a) $1 + \frac{1}{2} + \frac{1}{4} + \dots$,

   (b) $9 - 3 + 1 - \dots$.

2. Find the sum of the series

   (a) $r + r^2 + r^3 + \dots$, $(|r| < 1)$,

   (b) $(1 - a)^{-1} + (1 - a)^0 + (1 - a)^1 + (1 - a)^2 + \dots$.

   For what values of a is the series convergent?

3. Write each of the following repeating decimals as an equivalent common fraction.

   (a) $0.\overline{5}$

   (b) $0.0\overline{62}$

   (c) $3.\overline{297}$

   (d) $2.6\overline{9}$

4. What distance will a golf ball travel if it is dropped from a height of 72 inches, and if, after each fall, it rebounds 9/10 of the distance it fell.

5. Solve the following equation for x :

   $\frac{2}{3} = 1 + x + x^2 + \dots$.

6. Solve the following equation for x :

   $\frac{1 + x}{x} = x + x^2 + x^3 + \dots$.

7. Solve for a_1 and r if

   $a_1 + a_1r + a_1r^2 + \dots = \frac{3}{2}$, and

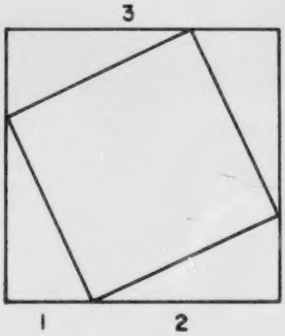
   $a_1 - a_1r + a_1r^2 - \dots = \frac{3}{4}$.

8. An equilateral triangle has a perimeter of 12 inches. By joining the midpoints of its sides with line segments a second triangle is formed. Suppose this operation is continued for each new triangle that is formed. Find the sum of the perimeters of all triangles including the original one.

9. A hare and a tortoise have a race, the tortoise having a 5000 yard handicap. The hare's speed is $V = 1000$ yards per minute; the tortoise's speed is $v = 1$ yard per minute. It is sometimes said that the hare can never catch the tortoise because he must first cover half the distance between them. Detect the fallacy.

10. A regular hexagon has a perimeter of 36 inches. By joining the consecutive midpoints of its sides with line segments, a second hexagon is formed. Suppose this is continued for each new hexagon. Find the sum of the perimeters of all hexagons including the original one.

11. A square has a perimeter of 12 inches. Along each side, a point is located one-third the distance to the right of each vertex. By joining consecutive points, a new square is formed. Suppose this process is continued for each new square.



    (a) Find the sum of the perimeters of all such squares.
    (b) Find the sum of the areas of all such squares.

13-7. Miscellaneous Exercises.

1. Find the sum of the series

$$\sum_{k=0}^{3} [(-2)^k - 2k] .$$

2. The following series are either arithmetic or geometric; continue each through 6 terms:

(a) $8 + 4 + 2 + \dots$,

(b) $3 + 6 + 9 + \dots$,

(c) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$,

(d) $\frac{1}{3} - 1 + 3 + \dots$,

(e) $a^2 + a^4 + a^6 + \dots$,

(f) $i - 1 - i + \dots$, $i = \sqrt{-1}$.

3. Find the sum of the series

$$\sum_{k=1}^{4} (2^k - k) .$$

4. Use the identity

$k = \frac{1}{2}[k(k + 1) - k(k - 1)]$ to give a new proof of Theorem 13-2a.

5. By use of the identity

$2k + 1 = (k + 1)^2 - k^2$, prove that

$$1 + 3 + 5 + \dots + (2n - 1) = n^2 .$$

6. By use of the equation

$$\frac{1}{k} - \frac{1}{k + 1} = \frac{1}{k(k + 1)} \text{ , show that}$$

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n + 1)} = \frac{n}{n + 1} \quad .$$

7. Show that the geometric mean between two positive integers is not greater than the arithmetic mean between them.

8. The harmonic mean between two numbers a and b is a number h whose reciprocal is the arithmetic mean between the reciprocals of a and b :

$$\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \text{ , } h = \frac{2ab}{b + a} \text{ .}$$

Show that in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

each term after the first is the harmonic mean between its two neighbors.

9. Show that the number of vertices of a cube is the harmonic mean between the number of its faces and the number of its edges.

10. Show that geometric mean between two numbers is also the geometric mean between their arithmetic and their harmonic means. First try the result for 2 and 8 .

11. Find the sum of the following series correct to 2-place decimals.

$$2\sqrt{3} \sum_{k=0}^{5} \frac{1}{(-3)^k (2k + 1)} \text{ , where } 2\sqrt{3} \approx 3.464 \text{ .}$$

12. Twenty-five stones are placed on the ground at intervals of 5 yards apart. A runner has to start from a basket 5 yards from the first stone, pick up the stones and bring them back to the basket one at a time. How many yards has he to run altogether?

13. Find the sum of n terms of the series whose r^{th} term is $\frac{3}{4}(3r + 1)$.

14. Two hikers start together on the same road. One of them travels uniformly 10 miles a day. The other travels 8 miles the first day and increases his pace by half a mile each succeeding day. After how many days will the latter overtake the former?

15. How many terms of the sum $1 + 3 + 5 + \ldots$ are needed to give 12321 ?

16. Find s_{20} if $a_3 = 5$, and $a_{37} = 82$.

17. (a) Find the sum of all even integers from 10 to 58 inclusive.
    (b) Find the sum of all odd integers from 9 to 57 inclusive.

18. A person saved thirty cents more each month than in the preceding month and in twelve years all of his savings amounted to $9,424.80 . How much did he save the first month? The last month?

19. If four quantities form an arithmetic sequence, show that
    (a) The sum of the squares of the extremes is greater than the sum of the squares of the means.
    (b) The product of the extremes is less than the product of the means.

20. (a) A constant c is added to each term of an arithmetic progression. Is the new series also an arithmetic progression; if so, what is the new difference and how is the new sum related to the original sum?

(b) If each term of an arithmetic progression is multiplied by a constant c is the new series an arithmetic progression; if so, what is the new difference and how is the new sum related to the original?

(c) A new series is obtained by adding a constant c to each term of a geometric progression. Is the new series a geometric progression; if so, what is the new ratio and how is the new sum related to the old?

(d) A new series is obtained by taking the reciprocal of each term of an arithmetic progression. Is the new series an arithmetic progression? What is the new difference?

(e) A new series is obtained by taking the reciprocal of each term of a geometric progression. Is the new series a geometric progression? What is the new ratio?

(f) Do the negatives of each term of a geometric progression form a geometric progression? If so, what is the new ratio?

21. Find the value of $\frac{1}{2n+1} \sum_{k=1}^{2n+1} k$.

22. Prove that if $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in arithmetic progression, then a^2 , b^2 , and c^2 are in arithmetic progression. (The converse is also true.)

23. Find the sum of $\frac{1}{1+\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1-\sqrt{x}} + \dots$ to n terms.

Hint: rationalize the denominators.

24. If the sum of an arithmetic progression is the same for m terms as for n terms, $m \neq n$, show that the sum of $m + n$ terms is zero.

*25. The sum of m terms of an arithmetic progression is n , and the sum of n terms is m . Find the sum of $m + n$ terms. $(m \neq n)$.

Chapter 14

PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM

14-1. Introduction. Counting Problems.

The process of counting involves three fundamental ideas.

(I_1) The first is that of a pairing, or one-to-one correspondence. Thus we count our fingers, or our guests at dinner, by associating with each one of the things being counted one of the natural numbers beginning with 1 and taking them "in order". We stop this process when we run out of fingers, guests, or whatever it is we are counting.

(I_2) The second idea is that underlying addition. Given two finite sets sharing no elements, the number of elements in their union is the sum of the number of elements in each. Thus the number of people at a swimming party is the sum of the number in the pool and the number not in the pool.

(I_3) The third idea is that underlying multiplication. Given n sets (where n is a natural number) no pair of which share any elements and each one of which may be paired with the set $\{1, 2, \ldots, m\}$ of all natural numbers not exceeding the natural number m, the number of elements in the union of the n given sets is $n \times m$. Thus we may count the students (or the seats) in a classroom by multiplying the number of rows by the number in each row (provided each row has as many as any other row).

To illustrate these ideas we present a method (involving all three of them) for proving

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2} .$$

(This formula was discussed in Chapter 13.)

Let us consider a collection of dots arranged in n rows, each containing $n + 1$ dots (Figure 14-1a). We pair the n rows to the natural numbers $1, 2, \ldots, n$ and the $n + 1$ columns to the natural numbers $1, 2, \ldots, n, n + 1$; and write the number associated with each row at its left, that associated with each column above. This is a use of the first idea, I_1. Now we draw a line across the array of dots as shown, dividing it into two parts.

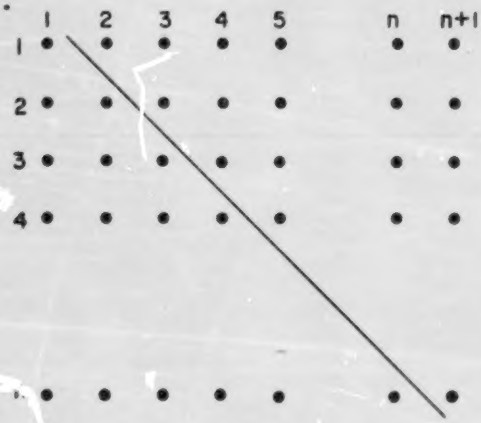


Fig. 14-1a

In that part below the line, we find that

the 1^{st} row has 1 dot ,
the 2^{nd} row has 2 dots ,
the 3^{rd} row has 3 dots ,
--- --- --- --- - ----
the n^{th} row has n dots .

Thus, since no two rows share any dots, our second fundamental idea, I_2, asserts that there are

$$1 + 2 + \ldots + n$$

dots below the line. Call this number s. Then

$$s = 1 + 2 + \ldots + n .$$

Now, above the line, we find that

the 2^{nd} column has 1 dot ,
the 3^{rd} column has 2 dots ,
the 4^{th} column has 3 dots ,
--- --- ------ --- - ----
the $(n + 1)^{st}$ column has n dots .

Hence there are also s dots above the diagonal.

Now, applying our second idea to these two parts of our array we find that $s + s$ is the total number of dots in it.

However, our third idea, I_3 , tells us that the total number of dots is $n(n + 1)$, the number of rows times the number of columns. Combining the results of our two counting methods, we have

$$2s = n(n + 1)$$

or

$$s = \frac{n(n + 1)}{2} .$$

Thus

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2} .$$

In this chapter we study a number of counting problems--i.e., problems whose solution may be made to depend on the three fundamental ideas of counting. Such problems occur frequently in mathematics, science, social studies and many other fields. One of the richest sources of these problems is the theory of games of chance. How many ways may one draw a straight flush or a full house from a fair deck, or roll a seven with a pair of dice? A water molecule (H_2O) has three atoms, and therefore it is planar. But a sugar molecule ($C_{12}H_{22}O_{11}$) has 45 . How many ways may these 45 atoms be arranged in space, and, how many of these arrangements are chemically feasible? We won't answer all of these questions--certainly not the last one--but we shall study ways to handle a great many of this sort.

Relatively few of the examples we shall give and the exercises we shall set can be considered earth-shaking. The interest will always be in the theory, in the methods, and in the ideas they illustrate, and only very rarely, in the "practical value" of their answers.

One of our objects in developing this theory is to obtain results which are "general" in the sense that the numbers involved are arbitrary. Of course, when one uses any of our results--for example, to answer questions about batting orders, snake-eyes, molecular structure, royal flushes, or how to line up a firing squad--he is dealing with a problem which comes with definite numbers of the things involved. The value of our general theory is that it can cope with any number of such problems, *no matter what* numbers may be involved in each of them.

Given enough time (in some cases, millions of years) and a large, fast, computer (some of which cost millions of dollars, plus upkeep) one could solve many of these specific counting problems by listing all the possibilities and tallying them. Our object is just the opposite. We set for ourselves the task of working such problems without actually listing all the possibilities. For example, in Section 14-4, we determine the number of Senatorial "committees" that are theoretically possible. This number is so monstrous it would be entirely out of the question even to contemplate making a list of the committees. Very likely, there isn't enough paper in the world. But the answer to the question of the Senatorial committees--which depends on the number of Senators (100)--is little or nothing compared to the numbers in some of the counting problems which arise in connection with, say, a mole of gas, for there are 6.025×10^{23} molecules in a mole. Indeed the number of possible committees which can be formed among *any* group of people in the world is a triviality compared to the number of possibilities for the chemical reactions which might occur in a toy balloon.

Many counting problems are actually infinitely many problems expressed as one. Examples are how many ways may a natural number n be written as a sum of four squares, or as a product of natural numbers. The answers depend on n , which may be any natural number. Thus specific cases like n = 3 or n = 10 , etc., may be handled by enumeration. But getting a formula good for arbitrary n is another matter entirely. Many such problems are beyond all methods known at present. In this chapter we discuss a few of the known ones.

Exercises 14-1

1. Consider the following array

```
                  L
                L O
              L O G
            L O G A
          L O G A R
        L O G A R I
      L O G A R I T
    L O G A R I T H
  L O G A R I T H M
L O G A R I T H M S
```

Determine the number of ways one may spell LOGARITHMS starting with any one of the L's and moving either down or to the right to an O, then either down or to the right to a G, etc., ending with the S. (Hint: begin with the top two or three lines, and determine the number of ways to spell LO and LOG , then work with the first four lines, first five, etc., until you recognize a pattern in your answers.)

2. (a) Write an addition table for the numbers from 1 to 6 .
   (b) Using this table answer the following questions about honest dice games:
      (i) What number is one "most likely" to roll?
      (ii) Is one "more likely" to roll a power of 2 or a multiple of 3 ?
      (iii) Is one "more likely" to roll a prime or a non-prime?

14-2. Ordered m-tuples.

Suppose we wish to count the number of routes from A to C via B in Figure 14-2a. There are three paths from A to B (denoted by a,b,c) and four paths from B to C (denoted by w,x,y,z)

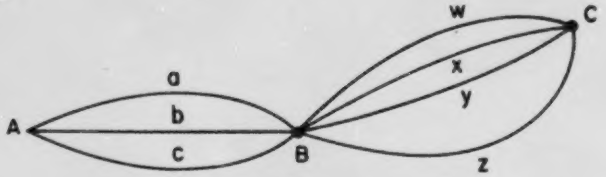


Fig. 14-2a

Now a route is completely described by naming a pair of these letters, provided we choose one from the set $\{a,b,c\}$ and the other from the set $\{w,x,y,z\}$. Thus (a,x) , (b,w) , (c,z) describe such routes. We tabulate all the possibilities in Table 14-2a.

	w	x	y	z
a	(a,w)	(a,x)	(a,y)	(a,z)
b	(b,w)	(b.x)	(b,y)	(b,z)
c	(c,w)	(c,x)	(c,y)	(c,z)

Table 14-2a

Each entry in the body of the table describes one of the possible routes. We see there are twelve of them.

Although we have enumerated all the cases in arriving at our answer, we may now see that this is quite unnecessary.

Our problem can be described as determining the number of *ordered pairs* which can be formed using an element of the set $\{a,b,c\}$ as first member and an element of $\{w,x,y,z\}$ as second member. (The body of Table 14-2a exhibits all of these pairs.) Since we are interested only in the *number* of pairs here we are interested merely in the *product* of the number of members in each of our sets--the number of *rows* in the table times the number of *columns*.

We may state the fundamental idea involved in a general way as follows.

> Given a pair of finite sets A_1 and A_2 with, respectively, n_1 and n_2 members each, there are $n_1 \times n_2$ ordered *pairs*, or *couples*, which may be formed with a member of A_1 as first member and a member of A_2 as second member. For our "route" problem, $A_1 = \{a,b,c\}$, $A_2 = \{w,x,y,z\}$, and $n_1 = 3$, $n_2 = 4$.

We illustrate this general principle by turning to a number of examples in which it may be used.

Example 14-2a: A quarter and a dime are tossed. How many head-tail pairs are possible?

Solution: There are two possibilities for the quarter $\{H,T\}$ and two for the dime $\{h,t\}$.

	h	t
H	(H,h)	(H,t)
T	(T,h)	(T,t)

The number of pairs is 4, the product 2×2.

Example 14-2b: A library contains 7 geometry books and 13 algebra books. How many ways may a student select two books, one of them a geometry and the other an algebra?

Solution: Here we are interested in the number of couples (geometry, algebra). According to our principle the number of such couples is 7×13, or 91 .

Example 14-2c: How many fractions may be formed whose numerator is a natural number not exceeding 10 and whose denominator is a natural number not exceeding 15 ? (Ignore the fact that some of these fractions represent the same rational number.)

Solution: $10 \times 15 = 150$.

Example 14-2d: Given a hundred men and a hundred women, it is possible to form 10,000 different couples--although not simultaneously!

Let us now extend our "route" problem by supposing there are five paths joining C to a fourth point D .

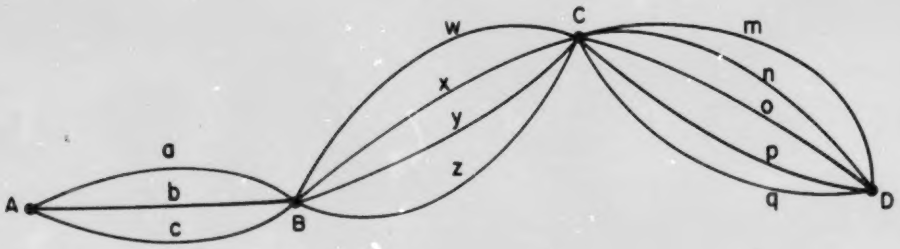


Fig. 14-2b

Let $\{m,n,o,p,q\}$ be the set of paths joining C and D . Now how many routes are available for a trip from A to D via B and C , using only the paths pictured?

We have already found there are twelve routes from A to C via B :

$$(a,w)\ ,\ (a,x)\ ,\ \ldots\ ,\ (b,w)\ ,\ \ldots\ ,\ (c,z)\ .$$

Taking advantage of this knowledge, we may describe any route from A to D via B and C by couples such as

$$((a,w),m)\ ,\ ((a,x),n)\ ,\ \text{etc.}$$

As before, we make a table.

	m	n	o	p	q
(a,w)	((a,w),m)				
(a,x)					
(a,y)				((a,y),p)	
(a,z)					
(b,w)					
(b,x)					
(b,y)		((b,y),n)			
(b,z)					
(c,w)					
(c,x)					
(c,y)					
(c,z)					((c,z),q)

Table 14-2a

We have indicated only a few of the entries in the body of the table. Using our principle we see at once that there are 12 × 5, or 60 , possibilities.

Since each of our new "couples" describes a route made up of three paths, we may drop the extra parentheses, writing simply (a,w,m) for ((a,w),m) , etc., and refer to (a,w,m) as an ordered triple. Thus each route from A to D via B and C in Figure 14-2b may be described by an ordered triple. Some more of these routes are (a,x,n) , (c,y,p) , (b,z,p) . Our principle tells us there are 60 ordered triples whose first component is an element of $\{a,b,c\}$, whose second component is an element of $\{w,x,y,z\}$, and whose third component is an element of $\{m,n,o,p,q\}$.

We may make further extensions to ordered quadruples, quintuples, etc.:

$$(a_1 , a_2 , a_3 , a_4) = ((a_1 , a_2 , a_3) , a_4)$$

$$(a_1 , a_2 , a_3 , a_4 , a_5) = ((a_1 , a_2 , a_3 , a_4) , a_5) ,$$

and generally to ordered m-tuples:

$$(a_1 , a_2 , \ldots , a_m)$$

with m components. Here m is any natural number.

As we saw in the case of ordered triples we may extend our general principle to covered ordered m-tuples:

> If $A_1 , A_2 , \ldots , A_m$ are finite sets having, respectively, $n_1 , n_2 , \ldots , n_m$ elements, there are $n_1 \times n_2 \times \ldots \times n_m$ ordered m-tuples of the form $(a_1 , a_2 , \ldots , a_m)$ where a_1 is a member of A_1 , a_2 a member of A_2 , ... , a_m a member of A_m .

Example 14-2e: In a certain club no member may run for more than one office at the same time. If in one election there are 8 candidates for president, 7 for vice-president, 4 for secretary, and 1 for treasurer, how many ways may these offices be filled?

Solution: We want the number of ordered quadruples (a_1 , a_2 , a_3 , a_4) where A_1 has 8 elements, A_2 has 7 , A_3 has 4 , and A_4 has 1 . Our principle tells us the answer is $8 \times 7 \times 4 \times 1$, or 224 .

Example 14-2f: Consider a club having 4 members, 3 offices, and a rule permitting any member to hold any number of these offices at the same time. How many ways may the offices be filled?

Solution: The number of ways of filling the offices is the number of ordered triples, each of whose components is any one of the members of the club. Here $m = 3$, $A_1 = A_2 = A_3$ and $n_1 = n_2 = n_3 = 4$. Our principles tells us the answer is $4 \times 4 \times 4$, or 64 .

Extending the result in Example 14-2f to cover the possibilities for m-tuples each of whose components are members of a set having n elements, we find there are n^m such m-tuples. For, in this case, we have

$$A_1 = A_2 = \dots = A_n$$

and

$$n_1 = n_2 = \dots = n_m = n \quad .$$

Thus

$$n_1 \times n_2 \times \dots \times n_m = n^m$$

gives the number of all possible m-tuples which can be formed, each of whose components belongs to a given set having n elements.

Exercises 14-2

1. A furniture company has twelve designs for chairs and five designs for tables. How many different pairs of table and chair designs can the company provide?
2. How many pitcher-catcher pairs may be formed from a set of four pitchers and two catchers?
3. How many pitcher-mug pairs may be formed from a set of eight pitchers and eleven mugs?

4. How many different committees consisting of one Democrat and one Republican may be formed from twelve Democrats and eight Republicans?
5. How many ways may a consonant-vowel pair be made using the letters of the word STANFORD ?
6. How many consonant-vowel pairs may be formed from the letters of the word COURAGE ?
7. How many numerals having two digits may be formed using the digits 1 , 2 , 3 , ... , 8 , 9 ?
8. Ten art students submitted posters in a contest which was to promote safety. How many ways could two prizes be awarded if one prize was to be given on the basis of the art work and the other on the basis of the safety slogan chosen?
9. There are four bridges from Cincinnati to Kentucky. How many ways may a round trip from Cincinnati to Kentucky be made if the return is not necessarily made on a different bridge?
10. How many ways may a two-letter "word" be formed from a twenty-six letter alphabet? (A "word" need not have meaning.)
11. How many different triples of Ace-King-Queen can be selected from a deck of 52 cards?
12. How many three digit numerals representing numbers less than 600 may be formed from the digits 1 , 2 , 3 , ... , 8 , 9 ?
13. Using the digits 1 , 2 , 3 , 4 , 6 , 8 , how many three digit numerals may be formed if the numbers they represent are even?
14. At a club election there are four candidates for president, four for vice-president, six for secretary, and six for treasurer. How many ways may the election result?
15. A freshman student must have a course schedule consisting of a foreign language, a natural science, a social science, and an English course. If there are four choices for the foreign language, six for the natural science, three for the social science, and two for English, how many different schedules are available for freshmen?

16. The Super-Super Eight offers twelve body styles, three different engines, and one hundred twenty color schemes. How many cars will a dealer need in order to show all possible cars?
17. How many quadruples Ace-King-Queen-Jack may be formed from a bridge deck?
18. From twelve masculine, nine feminine, and ten neuter words, how many ways are there to select an example consisting of one of each type?
19. In the decimal system of notation, how many numerals are there which have five digits?
20. A psychologist has rats running a maze having ten points at which the rat may turn left or right. How many ways could a rat run the maze if he followed a different route each time?

*21. Using the digits 3 , 5 , 6 , 7 , 9 how many three digit numerals representing numbers greater than 500 can be formed if (a) repetition of digits is allowed; (b) no repetition of digits is allowed?

*22. How many committees consisting of a Democrat and a Republican may be formed from five Democrats and eight Republicans if a certain Democrat refuses to work with either of two Republicans?

*23. There are five boys and eight girls at a dance. If Hepsibah and Prunella refuse to dance with either Hezy or Zeke, and Obediah will not dance with either Hepsibah or Cillisue, how many ways may dancing couples be paired?

14-3. Permutations.

Let A be the set {a,b,c} . We examine the ordered couples which may be formed using elements of A . We see there are two kinds: (i) those in which duplications occur, (ii) those without duplications. Thus (a,a) , (b,b) , (c,c) are of the first kind; the others of the second. There are 3 of the former--one for each element of A --and 6 , or $3^2 - 3$, of the latter.

	a	b	c
a	(a,a)	(a,b)	(a,c)
b	(b,a)	(b,b)	(b,c)
c	(c,a)	(c,b)	(c,c)

Table 14-3a

In general, given a set having n elements we may form n^2 ordered couples whose components are members of the given set. Of these n^2 couples, there are n (one for each element) which have duplications. Hence there are $n^2 - n$ without duplication.

Those ordered m-tuples of elements of a set having n elements which have no duplications are called permutations of the n elements taken m at a time or, for brevity, m - permutations of the set. Of course, $m \leq n$. Thus the number of 2-permutations of a set having n elements is $n^2 - n$.

The n-permutations of an n-element set are called simply permutations of the set.

There are many problems in mathematics, science, and other fields--including gambling--which may be solved with a knowledge of the number of m-permutations of an n-element set. We have determined this number for $m = 2$. We proceed to larger values of m .

As a preliminary, let us look again at the couples. We considered a table with n rows and n columns. To avoid duplications we omitted one couple from each row. Since we want only the number of 2-permutations, and not a list of them, it makes no difference in our counting problem if we simply remove a whole column from the complete table, rather than just one couple here-and-there in each row. Deleting one of the columns gives us a "reduced" table with n rows, $n - 1$ columns, and hence $n(n - 1)$ entries. This number checks with our previous "count" $n^2 - n$ and can be made to appear "plausible" if we think of the formation of ordered couples without duplication as a pair of "choices". We are free to choose any of the n elements as first component and any of the remaining $n - 1$ elements as second component. Since our "reduced" table has $n(n - 1)$ entries we can say that this pair of choices may be made in $n(n - 1)$ ways.

Moving on to 3-permutations of a set having n elements (i.e., ordered triples without duplication) we can imagine a table listing the 2-permutations on the left (there are $n(n - 1)$ of them) and the n elements of the set across the top. (For example, Table 14-3b, where $n = 3$.)

	a	b	c
(a,b)			(a,b,c)
(a,c)		(a,c,b)	
(b,c)	(b,c,a)		
(b,a)			(b,a,c)
(c,a)		(c,a,b)	
(c,b)	(c,b,a)		

Table 14-3b

To avoid duplications in the triples we must omit 2 triples from each of the n rows. As there are $(n^2 - n)n$ spaces in the table (number of rows times number of columns) and 2 blanks in each row, there are

$$(n^2 - n)n - 2(n^2 - n) ,$$

or $(n^2 - n)(n - 2)$, or $n(n - 1)(n - 2)$, entries in the table.

As with the couples, we are only interested in the number of triples. The same result may be obtained by simply deleting 2 columns, leaving $n(n - 1)$ rows and $n - 2$ columns. Hence we have another way of seeing that $n(n - 1)(n - 2)$ is the number of 3-permutations of a set having n elements.

Carrying on the same reasoning we may move to quadruples, quintuples, ... , m-tuples, and we get

$$n(n-1)(n - 2)(n - 3)$$

$$n(n - 1)(n - 2)(n - 3)(n - 4)$$

------- ----- ----- -----

$$n(n - 1)(n - 2)(n - 3)\dots(n - (m - 1)) ,$$

respectively, for the number of each having no duplication.

A great variety of symbols is used to denote the number of m-permutations of a set having n elements. Some of the more popular ones are

$$n^{(m)} ,\ {}_nP_m ,\ P_n^m ,\ P(n,m) .$$

We shall use the last of these in this book.

Writing $P(n,m)$ for the number of m-permutations of an n element set, our result may be expressed by the formula

$$P(n,m) = n(n - 1)(n - 2)\dots(n - m + 1) .$$

When $m = n$, we have

$$P(n,n) = n(n - 1)(n - 2 \dots(n - n + 1)$$

$$= n(n - 1)(n - 2 \dots 2 \cdot 1 .$$

The last product occurs so frequently in these and other problems, a special notation has been introduced for it:

$$n! = n(n - 1)\ldots 3 \cdot 2 \cdot 1 .$$

The expression " $n!$ " is read "n factorial" .

As examples of "factorials" we have

$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$
$$10! = 3,628,800$$

Observe the following property of factorials:

$$n! = n(n - 1)! \quad \text{or} \quad (n + 1)! = (n + 1)(n!) .$$

With this formula we may calculate $n!$ for each natural number n by a step-by-step process. However, since the numbers grow so fast these calculations soon get too involved. Recourse to tables is recommended. (At the end of Section 14-4 we give a table of the common logarithms of $n!$ for n up to 100 . This table will be useful for many of the computations arising in the next section. But it may also be used, in conjunction with the logarithm table following it, to get approximations to $n!$ for n up to 100 .)

The equation

$$(n + 1)! = (n + 1)n!$$

suggests the possibility of extending the definition of $n!$ to include $n = 0$:

$$1! = 1(0!) ,$$

i.e.

$$0! = 1 .$$

We shall find that doing this will enable us to simplify many of the problems we consider in this chapter.

Using factorial notation, our formula for $P(n,m)$ may be expressed quite compactly:

$$P(n,m) = n(n-1)(n-2)\ldots(n-m+1)$$

$$= n(n-1)(n-2)\ldots(n-m+1)\frac{(n-m)(n-m-1)\ldots 3\cdot 2\cdot 1}{(n-m)(n-m-1)\ldots 3\cdot 2\cdot 1}$$

$$= \frac{n!}{(n-m)!}\ .$$

Interpreting $0!$ to be 1, this expression holds even when $m = n$, for then

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!\ ,$$

which agrees with our previous expression for $P(n,n)$.

Example 14-3a: In a contest with twelve entries, how many ways can a first and a second prize be awarded if no entry is entitled to more than one prize?

Solution: Our problem calls for the number of couples of the form (one entry, another entry) without duplication, where the first entry wins first prize and the other wins second prize. The number of such couples is $P(12,2)$, so the answer is 12×11 or 132.

Example 14-3b: A map of four countries is to be colored with a different color for each country. If six colors are available, how many different ways may the map be colored?

Solution: We want the number of 4-permutations of a set having six elements. Each quadruple has the form (color of first country, color of second, color of third, color of fourth). The answer is $P(6,4)$, which is $6 \times 5 \times 4 \times 3$ or 360 ways.

Example 14-3c: Suppose a class of twenty students decides to leave the room in a different order each day. How many days would be required for the class to leave the room in all possible orders?

Answer: $20!$ days. If they work at it 365 days a year, it will take approximately 6.7 quadrillion years. ($20! \approx 2.4329 \times 10^{18}$ Even if they went through the door in a different order every second, it would take over 70 billion years.

Example 14-3d: How many ways may the numbers be arranged on a roulette wheel? (There are 38 "numbers": 00, 0 , and the natural numbers 1 , 2 , 3 , ... 36 .)

First Solution: If it is an honest wheel we cannot distinguish any one place from any other. Thus no matter where 00 may be placed, there are $37!$ ways of arranging the numbers 0 , 1 , 2 , ... , 36 . (If it is not an honest wheel, so that the places are distinguishable, the number is $38!$.)

Second Solution: Let us consider the set of all 38-permutations of $\{00, 0, 1, 2, 3, \ldots, 36\}$. Their number is $P(38,38) = 38!$. Corresponding to each such 38-permutation,

$$(a_1, a_2, \ldots, a_{38}),$$

there are 37 other permutations

$$(a_2, a_3, a_4, \ldots, a_{38}, a_1),$$

$$(a_3, a_4, \ldots, a_{38}, a_1, a_2),$$

-- --------------------------

$$(a_{38}, a_1, \ldots, a_{35}, a_{36}, a_{37}),$$

which cannot be distinguished from it on the (honest) wheel.

Thus if N is the number of "distinguishable" permutations of $\{00, 0, 1, \ldots, 36\}$, we have

$$38N = P(38,38) = 38!,$$

so

$$N = 37!$$

In the general case, there are $(n-1)!$ circular permutations of a set having n elements.

Exercises 14-3

1. How many five letter "words" may be formed from the letters A, B, C, D, E, F, and G ? How many if no letter is repeated?
2. How many ways may a president, vice-president, and secretary be elected from a club of twenty-five members if any member may hold any one of the three offices, but no member may hold more than one office simultaneously?
3. How many three digit numerals may be formed using the digits 1 , 2 , 3, 4, 5 , 6 if no digit is repeated in a numeral? How many if repetitions are allowed?
4. How many four digit numerals may be formed using the digits 1 , 2 , 3 , 4 , 5 , 6 if no digit is repeated in a numeral? How many if repetitions are allowed?
5. How many four digit numerals may be formed using the digits 2 , 4 , 6 , 8 if no digit is repeated in a numeral? How many if repetitions are allowed?
6. How many seven letter "words" may be formed using the letters of the word STANFORD ? How many if no letter is repeated?
7. How many different arrangements may be made for seven books on a shelf if the books are each of a different size?
8. Four persons are to ride in an airport limousine having six empty seats. How many different ways could they be seated?

9. Three traveling salesmen arrived at a town having four hotels. How many ways could they each choose a different hotel?
10. How many different combinations may be set on a lock having twenty numerals if the combination is a 3-permutation?
11. How many different batting orders may a baseball team manager form if he does not consider changing any but the last three places in the order?
12. How many different ways may the letters a, b, c, d, e, f be arranged with no repetitions so as to begin with ab in each case?
13. How many three digit numerals having no repeated digits may be formed from the digits 1 , 2 , 3 , 4 , 5 so that the middle digit is 3 ?
14. How many 5-permutations including the letter C may be formed from the letters A, B, C, D, E, F, G ?
15. How many ways may a photographer arrange four women and five men in two rows if women must stand in the first row and men in the second?
16. How many license plates may be made using two letters of a twenty-six letter alphabet followed by a four digit numeral? (Zero may be used at any place in the numeral.)
17. How many ways are there for eight children to form a ring around a May Pole?
18. If the number of ways to lay a set of tire weights in a line is six times the number of ways they may be placed on the tire rim, how many weights are there?
19. How many ways could King Arthur and eight of his knights sit at the Round Table if one of the seats was a throne chair for King Arthur only and there are eight other seats?

20. Find the exact numerical value of each of the following:

(a) $\frac{11!}{10!}$

(b) $\frac{8!}{5!}$

(c) $\frac{15!}{12!}$

(d) $\frac{52!}{50!2!}$

(e) $\frac{7!}{5!2!}$

(f) $\frac{15!}{12!3!}$

(g) $\frac{6!}{2!3!}$

(h) $\frac{2(3!)}{(2 \cdot 3)!}$

(i) $5(5! + 4!)$

21. Solve for the natural number n :

(a) $\frac{n!}{(n - 2)!} = 930$, $2 < n$.

(b) $P(n,5) = 20 \cdot P(n,3)$.

(c) $P(n + 2,4) - 72 \cdot P(n,2) = 0$.

(d) $P(n + 1,3) - 10 \cdot P(n - 1,2) =$

22. Simplify (n and m are natural numbers):

(a) $\frac{(n + 3)!}{n!}$

(b) $\frac{n!}{(n - 3)!}$

(c) $\frac{(n + 2)!}{(n - 1)!}$

(d) $\frac{(n - 2)!}{n!}$

(e) $(n - m - 2)!(n - m - 1)(n - m)$, $m \leq n$.

(f) $n[n! + (n - 1)!]$, $1 < n$.

(g) $\frac{1}{(n - 1)!} + \frac{1}{n!}$ $1 < n$.

(h) $\frac{n[n! + (n - 1)!]}{(n + 1)! - n!}$

(i) $\frac{(n + 2)! + (n - 1)!\,(n + 1)}{(n - 1)!(n + 1)}$, $1 < n$.

23. Prove each of the following for natural numbers m and n :

(a) $P(n,3) + 3 \cdot P(n,2) + P(n,1) = n^3$.

(b) $(n + 1)[n \cdot n! + (2n - 1)(n - 1)! + (n - 1)(n - 2)!]$
$= (n + 2)!$, $1 < n$.

(c) $P(n + 1,m) = (n + 1) \cdot P(n,m - 1)$, $m \leq n + 1$.

(d) $P(n,m) = m \cdot P(n - 1, m - 1) + P(n - 1,m)$, $m \leq n$.

(e) $P(n,m) = P(n - 2,m) + 2m \cdot P(n - 2,m - 1) + m(m - 1)$
$\cdot P(n - 2,m - 2)$, $m \leq n - 2$.

14-4. Combinations.

In this section we consider the following counting problems:

(i) Given a finite set having n elements, how many subsets does it have?

(ii) Given a finite set having n elements, how many 1-element, 2-element, 3-element, ... , m-element subsets does it have? (Here m is any natural number not exceeding n .)

Problem (i), being the easier, we consider first. Suppose our set has the elements

$$a_1 , a_2 , \dots , a_n .$$

The various subsets of $\{a_1 , a_2 , \dots , a_n\}$ may be formed by going down the list of members and for each member either taking it or not taking it. The process of forming a subset of $\{a_1 , a_2 , \dots , a_n\}$ can therefore be described by giving an ordered n-tuple, each of whose components is either T (meaning "take") or D (meaning "don't take").

For example, with $n = 4$, our set is $\{a_1 , a_2 , a_3 , a_4\}$ and the quadruple (T , T , D , D) yields the subset $\{a_1,a_2\}$:

$$\begin{array}{c} \{a_1 , a_2 , a_3 , a_4\} \\ (T , T , D , D) \\ \{a_1 , a_2 \qquad\qquad\} . \end{array}$$

The quadruple (D , T , D , T) gives $\{a_2 , a_4\}$:

$$\begin{array}{c} \{a_1 , a_2 , a_3 , a_4\} \\ (D , T , D , T) \\ \{\qquad a_2 , \qquad a_4\} . \end{array}$$

That each subset is described by such a "list of instructions" is illustrated by the following scheme. Given the subset $\{a_3 , a_4\}$ we have

$$\begin{array}{c} \{a_1 , a_2 , a_3 , a_4\} \\ \{ a_3 , a_4\} \\ (D , D , T , T) . \end{array}$$

Since each subset corresponds to exactly one of these ordered n-tuples, the number of subsets of $\{a_1 , a_2 , \ldots , a_n\}$ is the same as the number of ordered n-tuples one can form from the elements of the set $\{T , D\}$. At the end of Section 14-2, we found that the number of such ordered n-tuples is 2^n. Thus we have the following theorem.

Theorem 14-4a: There are 2^n subsets of a finite set which has n elements.

Note, in particular, two "subsets" which have been counted. They are the extreme cases in which the n-tuple has all T's (the "subset" corresponding to this n-tuple is the whole set); and the case in which the n-tuple has all D's (the "empty" or "void" subset, containing none of the members of the given set).

Example 14-4a: Since there are 100 Senators, the total number of Senate Committees which can be formed is $2^{100} - 1$, if we include the committee of the whole but exclude the committee with no members. This number is

$$1,267,650,600,228,229,401,496,703,205,375 .$$

We now consider Problem (ii): Given a finite set having n elements, how many m-element subsets does it have, where m is any natural number not exceeding n? An m-element subset of a set having n elements is often called a combination of the n elements taken m at a time.

Let us look at some examples. Given the set $\{a,b,c\}$ with three members, we have the following non-empty subsets

$$\begin{array}{ccccc} \{a\} & , & \{b\} & , & \{c\} \\ \{b,c\} & , & \{a,c\} & , & \{a,b\} \\ \{a,b,c\} & & & & \end{array}$$

This example is rather simple, but it tells us a good deal about the general question. Thus a set having n elements has n subsets each with 1 element (one such subset corresponding to each element--the one in the subset). A set having n elements has n subsets each with $n - 1$ elements (one such subset corresponding to each element--the one not in the subset). And, of course, there is only one n-element subset of a set having n elements; it is the whole set.

There are many different ways in use to denote the number of m-element subsets of a set having n elements. Some are

$$\binom{n}{m} , \ {}_nC_m , \ C_n^m , \ C(n,m) .$$

The last one of these we adopt in this book.

We have just seen that

$$C(n,1) = n , \ C(n,n - 1) = n , \ C(n,n) = 1 .$$

Now let us consider the subsets of $\{a,b,c,d\}$. Here $n = 4$. We already know $C(4,1)$, $C(4,3)$, $C(4,4)$. We have only to determine $C(4,2)$. The following scheme exhibits the 2-element subsets of $\{a,b,c,d\}$:

$$\begin{array}{llll} \{a\ , & b\ , & c\ , & d\} \\ \hline \{a\ , & b\} & & \\ \{a\ , & & c\} & \\ \{a\ , & & & d\} \\ & \{b\ , & c\} & \\ & \{b\ , & & d\} \\ & & \{c\ , & d\} \end{array}$$

There are six. Thus $C(4,2) = 6$.

We may detect a connection between our current problem and the permutation problems considered in Section 14-3 if we compare our last list with Table 14-4a exhibiting the ordered couples which may be formed with elements of {a,b,c,d} .

	a	b	c	d
a	(a,a)	(a,b)	(a,c)	(a,d)
b	(b,a)	(b,b)	(b,c)	(b,d)
c	(c,a)	(c,b)	(c,c)	(c,d)
d	(d,a)	(d,b)	(d,c)	(d,d)

Table 14-4a

The subsets

{a,b} , {a,c} , {a,d} ,
{b,c} , {b,d} ,
{c,d}

are represented in Table 14-4a by the couples

(a,b) , (a,c) , (a,d) ,
(b,c) , (b,d) ,
(c,d)

appearing in the upper right-hand corner. But they are also represented by the couples in the lower left of Table 14-4a:

(b,a) ,
(c,a) , (c,b) ,
(d,a) , (d,b) , (d,c) .

Let us match these couples to the subsets as follows

$\{a,b\}$:	(a,b) , (b,a)
$\{a,c\}$:	(a,c) , (c,a)
$\{b,c\}$:	(b,c) , (c,b)
$\{a,d\}$:	(a,d) , (d,a)
$\{b,d\}$:	(b,d) , (d,b)
$\{c,d\}$:	(c,d) , (d,c)

Examing this arrangement, we see that each ordered couple to the right of the line is a 2-permutation of the set at the left of its row. For each such 2-element subset, there are therefore $P(2,2)$ 2-permutations. Hence $P(2,2)$ is the number of columns to the right of the line. $C(4,2)$ is the number of rows. Since the total number of 2-permutations we can form from $\{a,b,c,d\}$ is $P(4,2)$, we have

$$C(4,2) \times P(2,2) = P(4,2) ,$$

and hence

$$C(4,2) = \frac{P(4,2)}{P(2,2)} = \frac{4 \cdot 3}{2 \cdot 1} = 6 .$$

We now consider a set having n elements. $C(n,m)$ denotes the (unknown) number of its m-element subsets. Let us imagine a table in which each of these m-element subsets determines a row. In each of the rows we write the $P(m,m)$ m-permutations of the subset which identifies the row. The total number of entries in this table is $P(n,m)$, the number of all m-element permutations of the given set. Multiplying the number of rows and the number of columns we have

$$C(n,m)\ P(m,n) = P(n,m)$$

or

$$C(n,m) = \frac{P(n,m)}{P(m,m)} = \frac{\frac{n!}{(n-m)!}}{m!} = \frac{n!}{m!(n-m)!}$$

Example 14-4b: Two cards are dealt from a deck of 52 cards. How many ways may this be done?

Solution: We want the number of 2-element subsets of a 52-element set.

$$C(52,2) = \frac{(52)!}{2!(50)!} = \frac{(52)(51)}{1 \cdot 2} = (26)(51) = 1326 \ .$$

Example 14-4c: How many 5-card poker hands containing the ace of spades are possible with a 52-card deck?

Answer: $C(51,4) = \frac{51!}{4!(47!)}$. Using the table for $\log n!$ (following these examples) we find

$$\begin{array}{rlrl} \log 4! = & 1.3802 & \log 51! = & 66.1906 \\ \log 47! = & \underline{59.4127} & \log 4!(47!) = & \underline{60.7929} \\ \log 4!(47!) = & 60.7929 & \log C(51,4) = & 5.3977 \end{array}$$

hence $C(51,4) \approx 2.5 \times 10^5$.

Example 14-4d: Show that $C(n,m) = C(n,n-m)$ and interpret this formula in terms of the subsets of a given set.

Solution: $C(n,n-m) = \frac{n!}{(n-m)!(n-(n-m))!} = \frac{n!}{(n-m)!m!}$

$$= C(n,m) \ .$$

$C(n,m)$ is the number of m-element subsets of a set having n elements. Each of these subsets may be paired with an $(n - m)$-element subset of the same set; namely, the subset containing none of the members of the original subset. This pairing shows that the m-element subsets and the $(n - m)$-element subsets of a given set are equally numerous; and that is exactly what the formula states.

Example 14-4e: How many ways may an arbitrary natural number n be represented as a sum of m natural numbers if we regard sums differing in the order of their terms as different "representations".

Solution: We look first at a special case: $n = 5$, $m = 3$. Let us consider 5 "tallies" in a row

$$| \ | \ | \ | \ | \ .$$

Our problem is equivalent to splitting this row of tallies into three parts. Thus

$$|*| \quad |*| \quad |$$

yields the sum $1 + 2 + 2$;

$$|*| \ ||*|$$

yields $1 + 3 + 1$, and

$$|*| * | \ | \ |$$

yields $1 + 1 + 3$. Splitting the row of tallies into three parts is accomplished by selecting 2 of the four spaces between adjacent tallies. This can be done in C(4,2), or 6 ways.

In the general case we have n tallies with $n - 1$ spaces:

$$| \underset{1}{\_} | \underset{2}{\_} | \underset{3}{\_} | \_ \cdots \underset{n-2}{\_} | \underset{n-1}{\_} | .$$

Each representation of n as a sum of m terms corresponds to a selection of an $(m - 1)$-element subset of the set of $n - 1$ spaces. The number is therefore

$$C(n - 1 , m - 1) .$$

Example 14-4f: If a class has 20 students and the classroom has 2 doors and 3 windows, how many different ways may the teacher and the students leave in case of a fire if at least one person goes through each of these exits?

Solution: Since it's "every man for himself", we treat all 21 souls on an equal basis. The natural number 21 may be written as a sum of 5 natural numbers (one term for each of the exits) in $C(21 - 1 , 5 - 1)$ ways, regarding as distinct such representations differing in the order of their terms.

$$C(21 - 1, 5 - 1) = C(20, 4) = 4{,}845 .$$

(The size of the answer justifies having a plan of egress ahead of time, obviating numerous hasty decisions.)

Example 14-4g: How many bridge hands of 13 cards contain exactly 5 spades?

Solution: We want the number of ordered couples (A, B), where A is a set of 5 spades and B is a set of 8 non-spades. There are $C(13,5)$ possibilities for A and $C(39,8)$ possibilities for B. Hence there are $C(13,5) \times C(39,8)$ such couples.

$$C(13,5) \times C(39,8) \approx 7.92 \times 10^{11} .$$

COMMON LOGARITHMS OF n!

n	log n!	n	log n!	n	log n!	n	log n!
0	0.0000	25	25.1907	50	64.4831	75	109.3946
1	0.0000	26	26.6056	51	66.1906	76	111.2754
2	0.3010	27	28.0370	52	67.9067	77	113.1619
3	0.7782	28	29.4841	53	69.6309	78	115.0540
4	1.3802	29	30.9465	54	71.3633	79	116.9516
5	2.0792	30	32.4237	55	73.1037	80	118.8547
6	2.8573	31	33.9150	56	74.8519	81	120.7632
7	3.7024	32	35.4202	57	76.6077	82	122.6770
8	4.6055	33	36.9387	58	78.3712	83	124.5961
9	5.5598	34	38.4702	59	80.1420	84	126.5204
10	6.5598	35	40.0142	60	81.9202	85	128.4498
11	7.6012	36	41.5705	61	83.7055	86	130.3843
12	8.6803	37	43.1387	62	85.4979	87	132.3238
13	9.7943	38	44.7185	63	87.2972	88	134.2683
14	10.9404	39	46.3096	64	89.1034	89	136.2177
15	12.1165	40	47.9117	65	90.9163	90	138.1719
16	13.3206	41	49.5244	66	92.7359	91	140.1310
17	14.5511	42	51.1477	67	94.5620	92	142.0948
18	15.8063	43	52.7812	68	96.3945	93	144.0633
19	17.0851	44	54.4246	69	98.2333	94	146.0364
20	18.3861	45	56.0778	70	100.0784	95	148.0141
21	19.7083	46	57.7406	71	101.9297	96	149.9964
22	21.0508	47	59.4127	72	103.7870	97	151.9831
23	22.4125	48	61.0939	73	105.6503	98	153.9744
24	23.7927	49	62.7841	74	107.5196	99	155.9700

FOUR-PLACE TABLE OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Exercises 14-4

1. Using the set $\{a, b, c, d\}$:
   (a) Find the number of subsets.
   (b) List the 3-element subsets.
   (c) List the 3-permutations for each of the 3-element subsets.
   (d) Find the value of $C(4,3)$.
2. Evaluate each of the following:
   (a) $C(10,2)$
   (b) $C(8,3)$
   (c) $C(12,5)$
   (d) $C(25,24)$
   (e) $C(12,10)$
   (f) $C(100,98)$
   (g) $\frac{C(8,5)}{3!}$
   (h) $\frac{C(26,21)}{P(26,5)}$
3. Calculate the value of $\log 100!$ to four decimal places.
4. A student is instructed to answer any eight of ten questions on an examination. How many different ways are there for him to choose the questions he answers?
5. There are ten entries in a round-robin tennis tournament. How many matches must be scheduled?
6. How many distinct lines are determined by fifteen points on a plane if no three of the points are collinear?
7. How many triangles are determined by eight points on a plane if no three of the points are collinear?
8. A seed company tests its tulip bulbs in sets of sixteen. Four bulbs are selected for planting from each set. If all four grow, the remaining twelve are sold with a guarantee that at least eight of them will grow. How many ways can the four bulbs be selected for test planting from a set of sixteen?
9. How many committees of four members may be formed from a set of nine possible members?
10. How many committees consisting of two Democrats and two Republicans may be formed from a set of seven Democrats and six Republicans?

11. How many parallelograms are determined by a set of eight parallel lines intersecting another set of five parallel lines?
12. A basketball squad consists of four centers, five forwards, and six guards. How many different teams may the coach form if players can be used only at their one position?
13. From a set of twenty consonants and the five vowels, how many "words" may be formed consisting of three different vowels and two different consonants if one of the vowels must be a ?
14. How many five letter "words" containing two vowels and three consonants may be formed from the letters of the word LOGARITHM ?
15. Referring to the array given in Exercise 14-1, 1, determine the number of ways one can spell LOGARITHMS starting from a given one of the L's, going right or down for the next letter each time, and ending at S. (Suppose the given L lies in the m^{th} row from the bottom; then it is necessary to move down $m - 1$ times between successive letters.) Check your result by using the formula you obtain to solve **Exercise** 14-1, 1 "again."

Using the table for $\log n!$ find approximate answers for Exercises 16, 17, ... , 22.

16. A sample of five items is to be selected from a set of one hundred. How many different samples may be formed?
17. How many different poker hands of five cards each can be formed from a deck of fifty-two cards?
18. How many samples of ten units may be formed from a set of one hundred light bulbs?
19. How many subsets of five cards containing exactly three aces may be formed from a deck of fifty-two cards?
20. How many bridge hands can have two six-card suits?
21. How many bridge hands have one seven-card suit and three two-card suits?
22. How many bridge hands have a "5-4-3-1" distribution?

23. If $C(n,12) = C(n,8)$, find the value of $C(n,17)$.

24. If $C(18,4) - C(18,m + 2) = 0$, find the value of $C(m,5)$.

25. Prove Pascal's Theorem: $C(n,m) = C(n - 1, m - 1) + C(n - 1,m)$, $1 \leq m \leq n - 1$.

26. Show that Pascal's Theorem may be illustrated by the following table (called Pascal's Triangle), where entries in the table are of the form $C(n,m)$ for $1 \leq m \leq n$, and extend the table through the line for $n = 10$.

n \ m	0	1	2	3	...		
1	1						
2	1	2	1				
3	1	3	3	1			
.	.	.	.	.	.		
.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.

27. Prove that $C(n,n - 2) = C(n - 1,n - 2) + C(n - 2,n - 3) + \ldots + C(2,1) + C(1,0)$, if $3 \leq n$.

14-5. The Binomial Theorem.

We are all familiar with the formula

$$(x + y)^2 = x^2 + 2xy + y^2 .$$

Higher powers of the binomial $x + y$ may be expressed as polynomials in x and y by multiplying each result in turn by $x + y$. Thus

$$\begin{aligned}(x + y)^3 &= (x + y)^2 (x + y) \\ &= (x^2 + 2xy + y^2)(x + y) \\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3 ,\end{aligned}$$

$$\begin{aligned}(x + y)^4 &= (x + y)^3 (x + y) \\ &= (x^3 + 3x^2y + 3xy^2 + y^3)(x + y) \\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 + x^3y \\ &\quad + 3x^2y^2 + 3xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 .\end{aligned}$$

Proceeding this way, we may derive the expansion of each higher power, $(x + y)^n$, in a step-by-step fashion.

However, it is possible to apply our theory of combinations to obtain the expansion of $(x + y)^n$ where n is an arbitrary natural number. Thus we may avoid the step-by-step process and write out the entire expansion for any given n without first determining the expansion for each smaller value of n . The saving, therefore, in calculation is very great. Suppose, for example, that you need to know the first 6 coefficients in the expansion of $(x + y)^{100}$. (For reasons we cannot explain here, such questions often arise in scientific and sociological problems.) Using the formula we shall derive, you would not have to find first all the coefficients in all the expansions up to $(x + y)^{100}$; the 6 coefficients you wanted could be written down without any preliminary calculations.

Before we attack the general problem, we recast it in a simpler form. Note that

$$(x + y)^n = [x(1 + \frac{y}{x})]^n = x^n(1 + \frac{y}{x})^n .$$

If we set $z = \frac{y}{x}$, our problem amounts to determining the expansion of $x^n(1 + z)^n$. This can be done if we determine the coefficients in the expansion of $(1 + z)^n$. For all we need do with this expansion is multiply each term by x^n . Finally replacing z by $\frac{y}{x}$ we can obtain the expansion of $(x + y)^n$.

We turn now to the expansion of $(1 + z)^n$. In order to obtain the coefficients in this expansion, we shift our attention to the product

$$(1 + z_1)(1 + z_2)\ldots(1 + z_n) .$$

Note that when $z_1 = z_2 = \ldots = z_n = z$, the product reduces to $(1 + z)^n$ since it has n factors.

We look first at some examples. For $n = 2$, we have

$$(1 + z_1)(1 + z_2) = 1 + (z_1 + z_2) + z_1z_2 .$$

For $n = 3$:

$$\begin{aligned}(1 + z_1)(1 + z_2)(1 + z_3) = 1 + (z_1 + z_2 + z_3) + (z_2z_3 + z_1z_3 \\ + z_1z_2) + z_1z_2z_3 .\end{aligned}$$

For $n = 4$:

$$\begin{aligned}(1 + z_1)(1 + z_2)(1 + z_3)(1 + z_4) &= 1 + (z_1 + z_2 + z_3 + z_4) \\ &+ (z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4) \\ &+ (z_2z_3z_4 + z_1z_3z_4 + z_1z_2z_4 + z_1z_2z_3) \\ &+ z_1z_2z_3z_4 .\end{aligned}$$

Studying these examples gives the clue to the general pattern. For $n = 2$, consider the set $\{z_1z_2\}$. Its non-empty subsets are

$$\{z_1\} , \{z_2\} , \text{ and } \{z_1z_2\} ;$$

each of which corresponds to a term in the expansion:

$$\begin{matrix}\{z_1\} , & \{z_2\} , & \{z_1z_2\} , \\ z_1 , & z_2 , & z_1z_2 .\end{matrix}$$

Similarly for $n = 3$. We list the non-empty subsets of $\{z_1, z_2, z_3\}$ and the terms appearing in the expansion which correspond to them:

$$\begin{array}{ccccccc} \{z_1\}, & \{z_2\}, & \{z_3\}, & \{z_2,z_3\}, & \{z_1,z_3\}, & \{z_1,z_2\}, & \{z_1,z_2,z_3\}, \\ z_1, & z_2, & z_3, & z_2 z_3, & z_1 z_3, & z_1 z_2, & z_1 z_2 z_3 . \end{array}$$

The expansions themselves (at least for $n = 2$, $n = 3$) are the sums of the terms listed, plus the extra term "1". The same pattern holds in the case $n = 4$.

What happens in these cases, when z_1, z_2, z_3, etc., are all replaced by z?

The terms contributing the first power of z to the sum are those corresponding to 1-element subsets; those contributing the second power of z to the sum are those corresponding to 2-element subsets; etc. Thus the number of z's (and hence the coefficient of z in the expansion) is $C(n,1)$; the coefficient of z^2 is $C(n,2)$; and for $n > 2$, the coefficient of z^3 is $C(n,3)$. These observations are valid at least when $n = 2$, $n = 3$, $n = 4$. The binomial theorem asserts that this is the case for n an arbitrary natural number.

In the general case, expanding the product

$$(1 + z_1)(1 + z_2)\ldots(1 + z_n)$$

yields terms of the following forms:

1	
$z_1, z_2, \ldots, z_n$	$C(n,1)$
$z_1z_2, z_1z_3, \ldots, z_{n-1}z_n$	$C(n,2)$
$z_1z_2z_3, \ldots, z_{n-2}z_{n-1}z_n$	$C(n,3)$
-----------------------------	----------
$z_1z_2 \ldots z_n$	$C(n,n)$

At the right of each line we give the number of terms in the line. Replacing z_1 , z_2 , ... , z_n each by z yields the terms

1	
$z, z, \ldots, z$	$C(n,1)$
$z^2, z^2, \ldots, z^2$	$C(n,2)$
$z^3, z^3, \ldots, z^3$	$C(n,3)$
- - - - - - - - - - - - -	- - - -
z^n	$C(n,n)$

Adding these terms we have our expansion:

$$(1 + z)^n = 1 + C(n,1)z + C(n,2)z^2 + C(n,3)z^3 + \ldots + C(n,n)z^n$$

or, more compactly,

$$(1 + z)^n = \sum_{m=0}^{n} C(n,m)z^m ,$$

if we agree to write $C(n,0) = 1$.

Returning to our original question regarding the expansion of $(x + y)^n$, we have, putting $\frac{y}{x}$ in place of z :

$$\begin{aligned}(x + y)^n &= x^n(1 + \tfrac{y}{x})^n \\ &= x^n[1 + C(n,1)\tfrac{y}{x} + C(n,2)\tfrac{y^2}{x^2} + \ldots + C(n,n)\tfrac{y^n}{x^n}] \\ &= x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \ldots + C(n,n)y^n ,\end{aligned}$$

or

$$\begin{aligned}(x + y)^n &= x^n(1 + \tfrac{y}{x})^n \\ &= x^n \sum_{m=0}^{n} C(n,m) \frac{y^m}{x^m} \\ &= \sum_{m=0}^{n} C(n,m)x^{n-m}y^m .\end{aligned}$$

This is the binomial theorem.

Theorem 14-5a: If n is any natural number, and if x and y are any real (or complex) numbers, then

$$\begin{aligned}(x + y)^n &= \sum_{m=0}^{n} C(n,m)x^{n-m}y^m \\ &= x^n + C(n,1)x^{n-1}y + \dots + C(n,n)y^n .\end{aligned}$$

Example 14-5a:

$$\begin{aligned}(x + y)^5 &= x^5 + C(5,1)x^4y + C(5,2)x^3y^2 + C(5,3)x^2y^3 \\ &\qquad + C(5,4)xy^4 + C(5,5)y^5 . \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 .\end{aligned}$$

Example 14-5b:

$$\begin{aligned}(x^2 - 3\sqrt{y})^4 &= (x^2)^4 + 4(x^2)^3(-3\sqrt{y}) + 6(x^2)^2 (-3\sqrt{y})^2 \\ &\qquad + 4(x^2)(-3\sqrt{y})^3 + (-3\sqrt{y})^4 \\ &= x^8 - 12x^6\sqrt{y} + 54x^4y - 108x^2y\sqrt{y} + 81y^2 .\end{aligned}$$

Note that if we take $x = y = 1$ in Theorem 14-4a, we obtain

$$2^n = (1 + 1)^n = \sum_{m=0}^{n} C(n,m)1^{n-m}1^m = \sum_{m=0}^{n} C(n,m) .$$

Thus the sum of the number of

0-element subsets	(C(n,0))
1-element subsets	(C(n,1))
2-element subsets	(C(n,2))
- - - - - - - - - -	- - - - - -
n-element subsets	(C(n,n))

of a set having n elements is 2^n. Note that we have already seen that 2^n is the total number of subsets (including the empty set and the whole set) of a set with n elements. Thus the binomial theorem ties together our solutions of the two problems we considered in Section 14-4. (Cf. The solutions of Exercises 14-1,1 and 14-4,15 .)

Exercises 14-5

1. Find the expansion for each of the following:
   - (a) $(x - y)^4$
   - (b) $(x - y)^5$
   - (c) $(a + b)^5$
   - (d) $(a + b)^7$
   - (e) $(2u + v)^6$
   - (f) $(r - 2s)^8$
   - (g) $(\frac{1}{2} + z)^6$
   - (h) $(x - \frac{1}{2})^9$
   - (i) $(x^2 + x)^8$
   - (j) $(c^2 - 2cd)^9$
   - (k) $(x^{-1} + 2y^{-2})^6$
   - (l) $\left(\frac{2}{x^2} - \frac{3}{y^3}\right)^5$
2. (a) What is the sum of the a,b exponents in the k^{th} term in the expansion of $(a + b)^n$, n,k in N and $k \leq n$?
   - (b) How many terms are there in the expansion of $(a + b)^{73}$? In $(a + b)^n$, n in N ?
   - (c) Which term in the expansion of $(a + b)^{32}$ is the middle term?
   - (d) For which values of n will the expansion of $(a + b)^n$, n in N , have no middle term?
   - (e) Give the C(n,m) form of the coefficient of the twenty-first term in the expansion of $(a + b)^{35}$.
   - (f) Which terms in the expansion of $(a + b)^{72}$ have their coefficients equal to the coefficient of the thirty-first term?
   - (g) If the coefficients of the sixth and sixteenth terms in the expansion of $(a + b)^n$, n in N , are equal, what is the value of n ?

(h) If the coefficients of the fourth and sixteenth terms in the expansion of $(a + b)^n$, n in N , are equal, find the middle term in the expansion.

3. (a) Find the seventh term in the expansion of $(a + b)^{15}$.

(b) Find the fourth term in the expansion of $(x - 5)^{13}$.

(c) Find the twelfth term in the expansion of $(2x - 1)^{13}$.

(d) Find the middle term in the expansion of $(\frac{3}{x} + \frac{x}{3})^{10}$.

(e) Find the middle term in the expansion of $(\frac{1}{x} - x^2)^{12}$.

(f) Find the eighth term in the expansion of $(1 - \frac{x^2}{2})^{14}$.

(g) Find the term having b^7 as a factor in the expansion of $(a + b)^{10}$.

(h) Find the term having y^5 as a factor in the expansion of $(x^2 - y)^9$.

(i) Find the term having x^{14} as a factor in the expansion of $(\frac{2}{x} - x^2)^{10}$.

(j) Find the term having y^3 as a factor in the expansion of $(x - 2y^{\frac{1}{2}})^9$.

(k) Find the term NOT having a factor of x in the expansion of $(x^2 - \frac{1}{x})^{12}$.

4. Find the numerical value for each of the following to four decimal places:

(a) 1.02^4 (Hint: $1.02 = 1 + 0.02$)

(b) 1.02^{12}

(c) 0.98^{12}

(d) 2.01^{10}

(e) 1.98^{10}

(f) $(1 - i)^8$

(g) $(2 - i)^5$

(h) $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^7$

14-6. Arrangements and Partitions.

We have considered the permutations and combinations of elements of a given finite set. In this section we consider another type of counting problem, one whose solution can be based on our previous results.

Example 14-6a: How many distinguishable arrangements are there of the letters in the word "loon" ?

Solution: If the word were "loan" instead of "loon", the methods discussed in Section 14-3 would apply directly and give the answer $P(4,4)$, i.e., 24 . However, we may expect that the answer to the present problem is much smaller, since we have duplications. Thus the permutations "loan" and "laon" correspond to the indistinguishable arrangements "loon" and "loon" in our current problem. Suppose, indeed, that we consider a complete list of the 4-permutations of the set $\{l,o,a,n\}$. These permutations may be paired with one another as follows:

loan , laon;   olan , alon;   oaln , aoln;
nloa , nlao;   nola , nalo;   noal , naol;
anlo , onla;   anol , onal;   lnoa , lnao;
oanl , aonl;   lano , lona;   alno , olna .

In each of these pairs the letter "l" occupies the same place, the letter "n" occupies the same place, but the letters "o" , "a" are interchanged. Replacing each "a" here by an "o" we see that each of these pairs yields a pair of indistinguishable arrangements of the letters of "loon". Thus the number of arrangements of the letters of "loon" is just half the corresponding number for "loan". Our answer is therefore 12 .

This example provides the key to the solution of the general problem of determining the number of arrangements of a list containing repetitions.

Corresponding to each arrangement of the letters of "loon" we have P(2,2) permutations of {l,o,a,n} arising from permutations of {o,a} .

Consider the problem of counting the number of arrangements of the 12 letters of "divisibility". Here the letter "i" occurs 5 times, but each of the other 7 letters occurs just once. Now P(12,12) is the number of arrangements of the elements of a set having 12 elements such as {d,a,v,e,s,i,b,o,l,u,t,y} . Let us write A for the (unknown) number of different arrangements of the letters of "divisibility". Corresponding to each of these arrangements are P(5,5) permutations of the letters d,$\underline{a}$,v,$\underline{e}$,s,$\underline{i}$,b,$\underline{o}$,l, $\underline{u}$,t,y .

Hence

$$A \times P(5,5) = P(12,12)$$

and

$$A = \frac{P(12,12)}{P(5,5)} = \frac{12!}{5!} = 12 \times 11 \times 10 \times 9 \times 8 \ .$$

Suppose, in general, we have a list of n items, m of which are the same but no two of the remaining n - m are the same. For example

$$x\ ,\ x\ ,\ y\ ,\ z\ ,\ x\ ,\ u\ ,\ x\ ,\ v\ . \qquad (n = 8\ ,\ m = 4)$$

Corresponding to the given list, let us consider a second list in which the duplicated items are distinguished (say, by subscripts). In our example,

$$x_1\ ,\ x_2\ ,\ y\ ,\ z\ ,\ x_3\ ,\ u\ ,\ x_4,\ \ v\ .$$

The number of arrangements of the second list is P(n,n) . Each arrangement of the first list corresponds to P(m,m) arrangements of the second. If A is the number of distinguishable arrangements of the original list,

$$A \cdot P(m,m) = P(n,n)$$

so

$$A = \frac{n!}{m!} \ .$$

Consider the corresponding problem for a list of n items, m_1 of one kind, m_2 of a second m_3 of a third kind, etc., with

$$n = m_1 + m_2 + \ldots + m_k .$$

If A is the number of distinguishable arrangements, then

$$n_1!n_2! \ldots n_k! \, A = n! ,$$

so

$$A = \frac{n!}{m_1! \, m_2! \ldots m_k!} .$$

This number is written as

$$\binom{n}{m_1 , m_2 , \ldots , m_k} = \frac{n!}{m_1! \, m_2! \ldots m_k!} .$$

We note two special cases. If $m_1 = m$ and $m_2 = \ldots = m_k = 1$ as in the previous examples, we have

$$A = \frac{n!}{m!1! \ldots 1!} = \frac{n!}{m!} ,$$

so that our earlier formulas are special cases of the general formula. If $k = 2$ and $m_1 = m$, then $m_2 = n - m$ and

$$\binom{n}{m_1 , m_2 , \ldots , m_k} = \frac{n!}{m!(n - m)!} = C(n,m) = \binom{n}{m} .$$

Example 14-6b: How many distinguishable arrangements are there of the letters in MISSISSIPPI ?

Solution: There are 11 letters, 4 of one kind (I) , 4 of another (S) , 2 of a third (P) , and 1 other (M) . Our formula gives

$$\frac{11!}{(4!)^2 \, (2!)(1!)} = 34,650 .$$

Exercises 14-6a

1. How many different six digit numerals may be written using the digit 5 once, a 4 three times, and a 3 twice?
2. How many five symbol code "words" may be formed using three dots and two dashes?
3. How many distinct "words" may be formed as arrangements of the letters of PARALLELEPIPED ?
4. Suppose that on one dark day in a certain hospital, four sets of identical male twins, two sets of identical female twins, nine males (single births), and eleven females (single births) are born, and cheap ink is used on their name-tags. The next day (even darker) the ink fades away. How many ways is it possible to mix the children up? (Use log $n!$ table of Section 14-5 to approximate the answer.)
5. How many different ways may the letters of QUARTUS be arranged so that the letter u follows the letter q ?
6. How many different arrangements of the letters of PALLMALL may be formed so that all of the l's are not together?
7. How many different ways may the letters of QUISQUIS be arranged so that each q is followed by a u ?
8. How many three letter arrangements of the letters of SNOOP may be formed? (Hint: consider cases as to the three letter word having 0 , 1 , or 2 O's.)
9. How many different arrangements of four letters may be made from the letters of SPOOL ?

Partitions of a Set.

By a partition of a set A we mean a collection of subsets of A having the properties

(i) no pair of the subsets share any members,

(ii) the union of all the subsets is A .

Thus each element of A is in one and only one of the subsets. The subsets themselves are called cells of the partition.

Example 14-6c: The two sets:

(1) the set of even natural numbers,
(2) the set of odd natural numbers,

form a partition of the set of natural numbers. Its cells are the two sets listed. The three sets:

(1) the set of positive real numbers,
(2) the set of negative real numbers,
(3) the set whose only member is zero,

form a partition of the set of real numbers.

Example 14-6d: The set $\{a,b,c\}$ has the following partitions (and no others without empty cells):

$\{\{a\} , \{b\} , \{c\}\}$; $\{\{a,b\} , \{c\}\}$; $\{\{a,c\} , \{b\}\}$;

$\{\{c,b\} , \{a\}\}$; $\{\{a,b,c\}\}$.

Since a partition is a collection, or set, whose members are themselves sets, we are obliged to be rather generous with our brackets when writing partitions. In the interests of economy (of ink) and ease of reading, we introduce an alternate notation and write, for the partitions listed in Example 14-6d:

$$[a;b;c] , [a,b;c] , [a,c;b] , [c,b;a] , [a,b,c] ,$$

respectively. If we need speak only of the cells, without exhibiting the elements in them, we shall write--as usual--

$$\{A_1 , A_2 , \ldots , A_k\}$$

for the partition of A whose cells are $A_1 , A_2 , \ldots , A_k$. (Here $A_1 , A_2 , \ldots , A_k$ are certain subsets of A .)

When we consider k-permutations of the set

$$\{A_1 , A_2 , \ldots , A_k\}$$

we deal with ordered k-tuples such as

$$(A_1 , A_2 , \dots , A_k) ,$$
$$(A_2 , A_1 , \dots , A_k) ,$$
$$(A_k , A_1 , \dots , A_2) .$$

Each of the k-permutations of a given partition of a set into k cells will be called an ordered partition of the set. If in the ordered partition $(A_1 , A_2 , \dots , A_k)$ there are n_1 elements in A_1 , n_2 elements in A_2 , ... , n_k elements in A_k , we shall call this partition an $(n_1; n_2 ; \dots , ; n_k)$ partition.

The problem we now put is this: Given a finite set A , having n elements, how many $(n_1; n_2; \dots, n_k)$ partitions $(A_1 , A_2 , \dots , A_k)$ of A are there? Since A_1 has n_1 elements, A_2 has n_2 , ... , A_k has n_k . We have, in view of the defining properties (i) , (ii) of a partition

$$n_1 + n_2 + \dots + n_k = n .$$

We shall see that we have really solved this problem already. All we must do to see this is to rephrase it appropriately. First, however, we look at an example.

Example 14-6e: Some of the (3;2;2) partitions of {a,b,c,d,e,f,g} are

[a,b,c;d,e;f,g] , [a,b,d;c,e;f,g] , [a,b,d;c,f;e,g] .

There are, of course, many more. Notice, however, that

[a,c,b;d,e;f,g] , [a,b,d;e,c;g,f] , [d,a,b;c,f;g,e]

are, respectively, simply other ways of writing the same three partitions as before. Consider, for example, the first in each of these lists. Using the more elaborate notation, these are

({a,b,c} , {d,e} , {f,g}) and ({a,c,b} , {d,e} , {f,g}) .

But the sets {a,b,c} and {a,c,b} are equal for they have the same elements. Similarly for the other sets in each of these partitions.

This example gives away the secret! The various cells are unchanged if their elements are rearranged--so far as their relationship to the partition itself is concerned, elements in the same cell are "alike". Thus permuting the elements in a given cell has no effect on the partition itself. Hence each ordered partition

$$[a,b,c;d,e;f,g]$$

corresponds to $3! \times 2! \times 2!$ permutations of the whole set. Since there are $7!$ permutations of the given set there are

$$\frac{7!}{3!2!2!}$$

$(3;2;2)$ partitions of it.

We may explain this result another way. We want an ordered triple of subsets where there is no duplication of elements. We can "choose" the first cell (which has 3 elements) in $C(7,3)$ ways. Since no pair of cells may share any members, we have only $C(4,2)$ "choices" for the second cell (which has 2 elements). Finally there are $C(2,2)$ "choices" for the third cell. Altogether, such an ordered partition may be formed in

$$C(7,3) \times C(4,2) \times C(2,2)$$

or

$$\frac{7!}{3!4!} \times \frac{4!}{2!2!} \times \frac{2!}{2!0!}$$

or

$$\frac{7!}{3!2!2!}$$

ways.

In the general case, the number of $(n_1;n_2; \ldots ; n_k)$ partitions of an n-element set (where $n = n_1 + n_2 + \ldots + n_k$) is

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}\,.$$

Example 14-6f: There are ten entries in an elimination tennis tournament. How many ways may the first round of matches be scheduled?

Solution: We want the number of $(2;2;2;2;2)$ partitions of a set with 10 elements (the 10 entries). This number is $\frac{10!}{(2!)^5}$ or, approximately, 1.1×10^5.

Example 14-6g: Given a set having n elements, how many $(m;n - m)$ partitions does it have?

Answer: $\frac{n!}{m!\,(n - m)!}$. This is $C(n,m)$. That it should be $C(n,m)$ may be seen if we note that the first cell has m elements, and all the other elements--if any--are in the second. Thus the $(m;n - m)$ partitions of an n-element set are paired with the subsets of the given set.

Exercises 14-6b

1. Eight men attend a sales convention and find they are to be in four double rooms. How many ways may they be assigned to these rooms?
2. How many subcommittees of two, three, and three members may be formed from a committee of eight members if each committee member can be on one and only one subcommittee?
3. In how many ways can 10 indistinguishable blue tickets and 30 indistinguishable red tickets be distributed among 40 people if each person is to receive exactly one ticket? (Use log n! table of Section 14-4 to approximate the answer.)
4. How many ways are there to arrange eight coins in a row so there will be three heads and five tails showing?

5. How many sets of bridge hands can be dealt to four players from a fifty-two card deck? (Use log n! table.)
6. (a) In how many ways can 6 people be partitioned into three teams each consisting of two people?
   (b) In how many ways can 12 people be partitioned into four teams each consisting of three people?
   (c) Generalize.

14-7. Selections with Repetition.

Suppose you are in a store having n kinds of items and more of each kind than you can afford to buy. How many different selections of m items can you make?

This problem differs in two ways from the "arrangement" questions we have considered. For one thing, we no longer take account of the order in which the items are "selected". For another, we suppose that--from the point of view of our resources--the supply of each kind is unlimited. The last supposition is for the sake of simplicity; without it the problem is much more difficult.

We begin with $m = 1$. The answer here is just n, for if we may select only one item our selection reduces to selecting one of the n kinds. The number of ways is then $C(n,1)$.

For $m = 2$, the question is more interesting. The two items may be alike or they may be different. But in either case, their "order" of selection is irrelevant.

Let us pair each of the *kinds* (of which there are n) with the numbers $1, 2, 3, \ldots, n$. Our problem is then to determine the number of unordered couples of the numbers $1, 2, \ldots, n$. We look at the table of the *ordered* couples of pairs of elements of $\{1, 2, \ldots, n\}$.

	1	2	3	4	...	n
1	(1,1)	(1,2)	(1,3)	(1,4)	...	(1,n)
2	(2,1)	(2,2)	(2,3)	(2,4)	...	(2,n)
3	(3,1)	(3,2)	(3,3)	(3,4)	...	(3,n)
.	...	...	...	...	...	...
.	...	...	...	...	...	...
.	...	...	...	...	...	...
n	(n,1)	(n,2)	(n,3)	(n,4)	...	(n,n)

The couples on the diagonal are those representing the selection of 2 items of the same kind; those not on the diagonal, the selection of 2 items of different kinds. But each couple below the diagonal represents the same selection as one above the diagonal. Suppose we erase all couples below the diagonal. Then we have just one couple for each of the selections we want to count. Their number is given by

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad ;$$

and is therefore $C(n+1, 2)$.

Before we go on to the case $m = 3$, let us observe that we have counted the number of ordered couples (a,b) of the form $1 \le a \le b \le n$, i.e., whose first component does not exceed its second component. This is another way of saying we count the unordered couples which may be formed from a set of n elements.

For $m = 3$, we want the number of ordered triples (a,b,c) with $1 \le a \le b \le c \le n$. As when $m = 2$, the chain of inequalities rules out each of the permutations of these triples but one.

Suppose we have made one selection, say a, $1 \leq a \leq n$. We have still to make two more, with $a \leq b \leq c \leq n$. Selecting b,c is equivalent to selecting two numbers from the set

$$\{a,\ a+1\ ,\ \ldots\ ,\ n\}$$

which has $(n - a) + 1$ members. Thus for each selection of a, $1 \leq a \leq n$, there are $\binom{n-a+2}{2}$ ways to select b and c satisfying $a < b < c < n$. Since no pair of triples with different first components can be the same (the first component being the least component), our second fundamental idea (Section 14-1) tells us the total number of selections is given by

$$\begin{aligned} S(n,3) = {} & C(n+1,2) + C(n,2) + C(n-1,2) + \ldots + C(4,2) \\ & + C(3,2) + C(2,2). \end{aligned}$$

Using Pascal's Theorem (Exercise 14-4, 25)

$$C(3,2) + C(3,3) = C(4,3)$$

and the fact that $C(2,2) = C(3,3)$, we have

$$C(3,2) + C(2,2) = C(4,3)\ .$$

$$\begin{aligned} \text{Hence}\quad S(n,3) &= C(n+1,2) + C(n,2) + C(n-1,2) + \ldots + C(4,2) + C(4,3) \\ &= C(n+1,2) + C(n,2) + \ldots + C(5,3) \\ &\text{- -------- - ------ - --- - ------} \\ &= C(n+1,2) + C(n+1,3) \\ &= C(n+2,3) \end{aligned}$$

The pattern emerges:

$$\begin{aligned} S(n,1) &= C(n,1) \\ S(n,2) &= C(n+1,2) \\ S(n,3) &= C(n+2,3)\ . \end{aligned}$$

In general,

$$S(n,m) = C(n+m-1,m) \ .$$

The general formula may be obtained by carrying on the same line of reasoning we have used in the cases $m = 1$, $m = 2$, $m = 3$.

Example 14-7a: Suppose you have 5 apples to give to 3 teachers. How many ways can you do this?

Solution: Here $m = 5$ and $n = 3$, for you are to *select* the *teachers* receiving the 5 items. The answer is given by

$$S(3,5) = C(7,5) = \binom{7}{2} = \frac{7 \cdot 6}{1 \cdot 2} = 21 \ .$$

Example 14-7b: Suppose a millionaire has 50 heirs and legatees. If he cuts none of them off without a cent and has just one million dollars to bequeath (after taxes and legal fees), how many different wills could he write?

Solution: $n = 50$, $m = 10^8$ (cents) . The number of wills is therefore $\binom{100,000,049}{50}$, which is approximately 7.9×10^{196} . This represents quite a few decisions.

Example 14-7c: How many ways may the natural number n be written as a sum of m *non-negative* integers, if we distinguish between sums differing in the order of their terms. (Compare Example 14-4e.)

Solution: When we considered the problem of representing n as a sum of m *natural* numbers we selected (without repetition) $m - 1$ spaces between n tallies arranged in a row. Extending this idea, selecting these spaces *with repetitions* will give us sums with 0 as a term. Thus, for $n = 5$, $m = 4$

$$| \quad \overline{1} \quad | \quad \overline{2} \quad | \quad \overline{3} \quad | \quad \overline{4} \quad |$$

the selection (1,1,3) gives $5 = 1 + 0 + 2 + 2$. However to allow for the first and last terms being zero, we should introduce 2 more spaces: one before the first tally and another after the last:

$$\underset{1}{\underline{\quad}}\ \big|\ \underset{2}{\underline{\quad}}\ \big|\ \underset{3}{\underline{\quad}}\ \big|\ \underset{4}{\underline{\quad}}\ \big|\ \underset{5}{\underline{\quad}}\ \big|\ \underset{6}{\underline{\quad}}$$

Thus, now, $(1,1,3)$ represents

$$5 = 0 + 0 + 2 + 3$$

and $(1,3,6)$ represents

$$5 = 0 + 2 + 3 + 0 .$$

With these extra spaces, we now have $n + 1$ spaces in the general case, of which we are to select $m - 1$ allowing repetitions. The number of such selections is given by

$$S(n + 1 , m - 1) = C[(n+1) + (m-1) - 1, m-1] = C(n+m-1, m-1) .$$

Exercises 14-7

1. A post office has ten types of stamps. How many ways may a person buy twelve stamps?
2. How many ways are there to select five packages of cheese from a bin containing ten kinds?
3. A piggy bank is passed to five people who place in it one coin each. If the coins are pennies, nickels, dimes, quarters, half dollars, or silver dollars, how many sets of coins might there be in the bank, assuming it to be empty at the start?
4. If the faces of two dice are numbered 0 , 1 , 3 , 7 , 15 , 31 , how many different totals can be cast?
5. How many dominoes are there in a set ranging from double blank to double twelve?
6. Delete the last eleven words of Example 14-4f and answer the question thus formed.

14-8. Miscellaneous Exercises.

1. How many different arrangements may be formed from the letters of the word MADAM ?
2. How many committees of seven persons may be formed from a set of ten persons?
3. How many distinct lines are determined by twelve points on a plane if no three of the points are collinear?
4. How many diagonals can be drawn in a convex polygon of n-sides?
5. How many permutations of the letters of COMPLEX will end in X ?
6. How many of the 5-permutations of the letters A, B, C, D, E, F, G will have A at the beginning or at the end?
7. How many different ways may exactly three heads show in a toss of five coins?
8. How many ways are there to seat ten persons around a table, if a certain pair of persons must sit next to each other?
9. How many four digit numerals may be formed from the set {1 , 2 , 3 , ... , 8 , 9} if no digits may be repeated and the numbers they represent are odd.
10. How many "words" containing three consonants and two vowels may be formed from a set of ten consonants and the five vowels?
11. How many five letter "words" may be formed from a twenty-six letter alphabet if the first letter is not repeated, but repetitions may occur in the other four places?
12. How many arrangements of three men and three women may be made at a round table if the men and women must sit alternately?
13. If all possible pairs of numbers, repetitions of digits not permitted, are selected from the set {1 , 2 , 3 , 4, 6} , in how many cases will the sum be even?

14. A bag contains five red balls, four white balls, and three black balls. How many different ways may three balls be drawn if each ball is to be a different color?
15. How many different hands containing three queens and a pair may be formed from a deck of fifty-two cards?
16. How many different signals may be formed from two red flags and three blue flags if any four of the flags are hoisted on a flagpole in a vertical line and the flags differ only in color?
17. How many ways may three boys and three girls stand in line if no two boys stand next to each other and no two girls stand next to each other?
18. How many ways are there to arrange a set of fifteen different books by size on a shelf if five of them are large, seven are medium size, and three are small?
19. How many three digit numerals are there that do not contain the digits 5 or 0 ?
20. How many ways may nine hooks be clipped onto a steel ring?
21. How many ways are there to seat seven persons in a row if two of them will not sit next to each other?
22. If a set of six different books is used, how many ways could three or more of them be arranged on a shelf?
23. How many ways are there to form a dinner party for seven persons from a set of ten persons if a certain pair of the ten will not attend the same dinner party?
24. How many ways may four boys of unequal heights stand in a line if no boy stands between two taller ones?
25. How many 5-permutations of the letters a,b,c,d,e,f,g do not contain b ?
26. How many ways are there for a man to invite one or more of his six friends to his home?
27. Find the number of arrangements of the letters of BOULDER if no two vowels are together.

28. How many three digit numerals representing even numbers greater than 234 may be formed using the digits 1 , 3 , 4 , 5 , 6 , 8 , 9 with no repetitions of digits permitted?

*29. How many three digit numerals representing even numbers greater than 234 may be formed using the digits 1 , 3 , 4 , 5 , 6 , 8 , 9 ?

*30. How many three digit numerals representing even numbers greater than 234 may be formed using the digits 2 , 3 , 4 , 5 , 6 , 8 , 9 if repetitions of digits are permitted? If repetitions of digits are not permitted?

31. Suppose n tickets, numbered serially, are printed for a raffle. Suppose they are all sold and each purchaser counterfeits $(m - 1)$ copies of his stub and sneaks them into a bowl (so that each of the n numbers appears on m tickets in the bowl). Two prizes are to be awarded and hence two stubs must be drawn.

(a) How many ways is it possible to draw two stubs?

(b) How many of these ways result in both numbers being the same?

(c) The ratio of the answer in (b) to that in (a) indicates the chances of exposing one of the counterfeiters. Compute this ratio for each pair (n,m) with n,m in the ranges $1 < n \leq 5$, $1 < m \leq 5$, m,n in N .

(d) What conclusions do you draw concerning the risk of being caught if

(i) n increases for fixed m ,

(ii) m increases for fixed n ?

Chapter 15

ALGEBRAIC STRUCTURES

15-1. Introduction.

During our course of study of this book, we have met several number systems: the systems of the natural numbers, the integers, the rational numbers, the real numbers and the complex numbers. In each of these systems we saw that our concern was with the following:

(1) Objects or elements, here numbers;
(2) Two operations, addition and multiplication;
(3) Laws satisfied by these operations, such as the commutative and associative laws of addition and multiplication and the distributive law.

If we stop and reflect for a moment, we see that many of the algebraic computations which we carried out were independent of the nature of the numbers with which we were operating and depended solely on the fact that the operations in question were subject to laws respected in each system. Thus, for example, if we consider the identity

15-1a $$a^2 - b^2 = (a + b)(a - b)$$

and think of this assertion as applying to a and b taken as

(1) integers,
(2) rational numbers,
(3) real numbers,
(4) complex numbers,

we see that, if we established the Identity 15-1a at the earliest stage for integers and observed

(1) that the verification depended only on the distributive law, the associative laws and commutative laws and properties of the additive inverse, and

(2) that each of the laws and properties invoked were in force for the complex number system,

then it would be unnecessary to repeat the verification for the case where a and b are complex numbers.

Without such laws algebraic computation as we know it would cease to exist. The whole source of algebraic computation is to be found in these laws.

We can, if we like, seek to abstract what is algebraically essential and common to several specific number systems and develop algebraic results which hold for each of these systems without having to repeat our work in each special case. This approach is of great importance in many parts of modern mathematics, especially in modern higher algebra which is sometimes also called abstract algebra.

What is the nature of the fundamental algebraic operations which we have met? Let us take the addition of real numbers. We are given real numbers, say a and b, in order, or, if we like, the ordered pair (a,b). The operation of addition assigns to the ordered pair (a,b) a unique real number which we designate $a + b$. The words "assigns" and "unique" give the secret away. The operation of addition (of real numbers) is a function defined for each ordered pair of real numbers which assigns to each such ordered pair (a,b) of real numbers a real number, the sum $a + b$. It should be observed that while most of the functions which we have met assigned real numbers to real numbers, the function concept is an extremely general one and we may certainly consider a function f which assigns to each element a of a given class A a unique element (labelled $f(a)$) of a given class B. In the example of addition of reals, the class A

is the set of ordered pairs of real numbers and the class B is the set of real numbers itself. There is a point concerning notation that should be made. Instead of writing the real number associated with the ordered pair (a,b) in function notation, say S[(a,b)] , where S (standing for "sum") is the function just described, we use the usual notation and write $a + b$.

15-2. Internal Operation.

Let us try to abstract what is algebraically essential in the example of addition of real numbers. Suppose that A is an arbitrary non-empty set of elements, the nature of which need not concern us. Suppose further that there is given a function which is defined for the ordered pairs (a,b) , where $a \in A$ and $b \in A$, which assigns to each such ordered pair a member of A . Such a function is called an internal operation in A . (It is called "internal" because the components a and b of the input (a,b) are drawn from A and the output assigned by the function is also a member of A . Hence, the operation in question does not involve data taken outside of A .)

There is also a notion of an external operation and, indeed, an example is to be found in the algebra of vectors when one considers real multiples of a given vector so that input is an ordered pair of the form (real number, vector) and output is a vector. Here we go outside the domain of vectors to specify the input--hence "external".

However, in this chapter we shall consider only internal operations and for that reason we shall henceforth simply say "operation" rather than "internal operation". As it is customary, we shall usually denote an operation by a multiplication sign $\cdot$ and the element assigned to the ordered pair (a,b) by $a \cdot b$ when we are concerned with a single operation. We shall also write "ab" for "a · b" when there is no doubt about the meaning. We shall have occasion later to deal with two operations and then we shall usually use + and · to denote the two operations.

If we are concerned with *finite* sets A , we may specify with the aid of a multiplication table how a given operation acts in the same way that we listed the sum and product of certain important pairs of natural numbers with the aid of addition and multiplication tables in elementary arithmetic. The procedure is to use a square table marking rows by the elements of the set A and columns by the elements of the set A . The row markings are indicated at the left of the body of the table and the column markings are indicated above the body of the table. Given a,b $\in$ A , in the space in the body of the table belonging to the row marked "a" and the column marked "b" , we record the element associated with (a,b) by the operation · .

Here is a simple example: Let A = {0,1} and let · denote conventional multiplication in the real number system. Then the operation · may be tabulated as follows:

a \ b	0	1
0	0	0
1	0	1 .

Suppose that we consider a set A consisting of two distinct elements a and b and we ask in how many ways can we specify an operation in A . This amounts to constructing in all possible ways two-by-two square tables in each space of which is recorded an element of A . Here are some:

a \ b	a	b
a	a	a
b	a	a ,

a \ b	a	b
a	b	b
b	b	b ,

a \ b	a	b
a	a	a
b	a	b ,

a \ b	a	b
a	a	b
b	b	a .

There are 16 such operations in A .

Exercises 15-2

1. List the remaining 12 operations in A .
2. Let $A = \{1, i, -1, -i\}$ and let $\cdot$ denote conventional multiplication for complex numbers. Show that $\cdot$ is an operation in A and construct the table for $\cdot$

It is of interest to note that, if A is a finite set containing n elements, then there are n^{n^2} distinct operations in A . (For $n = 2$, we have $2^4 = 16$ distinct operations in A ; for $n = 3$, we have $3^9 = 19,683$ distinct operations in A .)

We shall be interested in studying the composite object consisting of a non-empty set A and one or two operations in A. Precisely, the term "composite object" is to be taken here to mean either an ordered pair of the form $(A, \cdot)$ where $\cdot$ is an operation in A or an ordered triple of the form $(A, +, \cdot)$ where + and $\cdot$ are operations in A . Such a composite object is called an algebraic structure with one operation (or two operations respectively). An example of a structure with one operation is given by taking A as the set of integers and $\cdot$ as the customary addition. An example of a structure with two operations is given by taking A as the set of real numbers and + and $\cdot$ respectively as the customary addition and multiplication for the reals. Another example of a structure with two operations is given by taking A as the set of real numbers, $\cdot$ as the customary multiplication for the reals, and + as the customary addition for the reals.

Now it turns out that the interesting structures are those which are subject to various laws. We saw that the number systems which we studied earlier were structures with two operations which respected such laws as the commutative laws, the associative laws, and the distributive law. If we wished to take into account structures which are not subject to any restrictions or laws, we would be faced with many different kinds of structures having very few properties in common. We cannot hope to find interesting results which would be valid for all structures with a given set A and with a given number of operations.

On occasion, instead of referring to the structure "(A, ·)" or "(A, + , ·)" we shall use the less formal "A together with the operation · " or "A together with the operations + and · " respectively, as well as "A and the operation · ", etc.

We shall concentrate on two important structures which permeate elementary algebra--the group and the field. Our interest will center principally on the notion of a field which embraces three of the important number systems which we have met so far--the systems of the rationals, the reals, and the complex numbers.

15-3. Group.

Suppose that we consider a structure with one operation (A, ·) . The example which we cited above, where A is the set of integers and · is the customary addition, has the following two properties:

(1) The associative law for addition is satisfied.

(2) Given integers a,b, there exists a unique integer x satisfying $a + x = b$ and there exists an integer y satisfying $y + a = b$.

(We ignore deliberately the question of the equality of x and y for a reason which will become clear presently.) If we ask for structures with one operation which share these listed properties with this special structure, we are led to the very important structures with one operation called groups. They appear throughout mathematics in many different guises. The study of groups as such is an instance of algebra at its purest.

Specifically $(A, \cdot)$ is said to be a group provided that the following two conditions are satisfied:

G 1. The operation $\cdot$ is associative. That is, given elements a, b, c in A, we have

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) .$$

G 2. Given elements a, b in A, each of the equations

$$a \cdot x = b$$

and

$$y \cdot a = b$$

has a unique solution in A.

It is to be observed that we have not required that the operation $\cdot$ be commutative. In fact, we shall meet examples where $\cdot$ does not satisfy the commutative law which asserts that $a \cdot b = b \cdot a$ for all $a, b \in A$. This is why it was important in defining the notion of operation to have as our input an ordered pair of elements of A. The order in which the components are assigned may very well be essential. If the operation $\cdot$ satisfies the commutative law, the group is called commutative or, as is more usual, abelian, in honor of the great Norwegian mathematician N. H. Abel (1802-1829) who did pioneer work in the theory of groups.

Let us consider some examples of groups drawn from our earlier experience. In these examples the operations are the standard ones of the number systems so that the groups in question are necessarily abelian. We shall consider an example of a non-abelian group later (Section 15-5).

Example 15-3a. A = set of integers; the operation $\cdot$ is the conventional addition + . The second postulate states that the equation $a + x = b$, where a and b are integers, has a unique integral solution.

Example 15-3b. A = set of real numbers different from zero; $\cdot$ is the conventional multiplication.

Example 15-3c. A = set of vectors in 3-space; $\cdot$ is the addition $\oplus$.

Exercises 15-3

1. Verify that each of the cited examples satisfies the group postulates G 1 , G 2 . Show that the following are also examples of groups:

   Example 15-3d. A is the set of n^{th} roots of 1, where n is a positive integer, and $\cdot$ is the conventional multiplication for complex numbers. Here it is to be observed that A has just n elements.

   Example 15-3e. A is the set of positive rational numbers; $\cdot$ is the conventional multiplication.

2. In what way does the following fail to yield an example of a group: A = set of all complex numbers and $\cdot$ is the conventional multiplication?
3. Let A denote the set of real numbers of the form $a + b\sqrt{2}$ where a and b are integers and let $\cdot$ be the conventional addition. Verify that $\cdot$ is an operation in A and that the group postulates are satisfied.
4. Let A denote the set of real numbers different from zero of the form $a + b\sqrt{2}$ where a and b are rational and let $\cdot$ be the conventional multiplication. Verify that $\cdot$ is an operation in A and that the group postulates are satisfied.

15-4. Some General Properties of Groups.

We have seen in our earlier work with number systems that an important role was played by the notions of additive identity, additive inverse, multiplicative identity, multiplicative inverse. The counterparts of these notions appear in general group theory as we shall now see. We must not forget that the commutative law need not be in effect for an arbitrary group!

Identity element. Here we are concerned with the question whether there is an element e in A which has the property that $a \cdot e = e \cdot a = a$ for all elements $a \in A$. In each of the cited examples of Section 15-3 there is precisely one element with this property. Thus in Example 1, the integer 0 is the unique element having the stated property; in Example 2, it is 1; in Example 3, it is the zero vector $(0,0,0)$; in Example 4, it is 1; in Example 5, it is 1. We now turn to the situation for an arbitrary group and a proof of the following theorem:

Theorem 15-4a: Given the group consisting of the set A and operation $\cdot$, there is a unique element e of A which satisfies the following condition:

$$a \cdot e = e \cdot a = a$$

for all $a \in A$.

(The element e is called the identity element of the group. Note how this is in agreement with earlier usage.)

Proof: We fix an element $b \in A$. That there is at most one element e having the stated property follows from the fact that e is a solution of the equation $b \cdot x = b$ which has precisely one solution.

Now let e denote the solution of $b \cdot x = b$ and let us verify that $a \cdot e = a$ for all a in A. Given $a \in A$, let c satisfy $c \cdot b = a$. That is, c is the unique solution of $y \cdot b = a$. Our reason for introducing c is that, if we write a as $c \cdot b$, we are in a position to relate the product ae (which we should like to show is equal to a) to the product

$b \cdot e$ about which we have information. Specifically,

$$\begin{aligned} a \cdot e &= (c \cdot b) \cdot e \\ &= c \cdot (b \cdot e) \\ &= c \cdot b \\ &= a \ . \end{aligned}$$

The proof of the theorem will be complete when we show that we also have $e \cdot a = a$ for all a in A. Given $a \in A$, let d denote the unique solution of the equation $y \cdot a = a$. In order to relate d and e, we introduce f the unique solution of the equation $a \cdot x = e$ (thereby linking the elements a and e). From $d \cdot a = a$ and $a \cdot f = e$, we have

$$\begin{aligned} (d \cdot a) \cdot f &= a \cdot f \\ &= e \ . \end{aligned}$$

From the associative law and $af = e$, we have

$$\begin{aligned} (d \cdot a) \cdot f &= d \cdot (a \cdot f) \\ &= d \cdot e \ . \end{aligned}$$

Taken together these equalities yield

$$d \cdot e = e \ .$$

Now e satisfies the equation $y \cdot e = e$. (Recall that $ae = a$ for all a in A, in particular for $a = e$. This yields $e \cdot e = e$.) Since e and d both satisfy the equation $y \cdot e = e$ and since this equation has a unique solution, $e = d$. Hence on taking account of the relation $d \cdot a = a$, we have $e \cdot a = a$. The proof of the theorem is now complete.

The notation "e" will be reserved for the identity element.

Inverse element. Given $a \in A$, let us consider the two equations

$$a \cdot x = e \quad \text{and} \quad y \cdot a = e \ .$$

Since we do not have the commutative law at our disposal, it is not obvious that the solutions x and y of these respective equations are equal. Let us see whether, in spite of the non-availability of the commutative law, $x = y$. Let us multiply each side of $a \cdot x = e$ *on the left* by y. We obtain

$$y \cdot (a \cdot x) = y \cdot e \ .$$

Using the associative law and the basic property of the identity, we obtain

$$(y \cdot a) \cdot x = y \ .$$

Hence

$$e \cdot x = y \ .$$

Since

$$e \cdot x = x \ ,$$

we conclude that $x = y$. The *common* solution of $a \cdot x = e$ and $y \cdot a = e$ is called simply the *inverse* of a. It is denoted a^{-1}.

Exercises 15-4

1. Determine the inverse element of an arbitrary element for each of the groups examined in Section 15-3. The answer is to be stated in terms of the special interpretation of a group given by the example. Thus in Example 15-3a, the answer is "the inverse of a is $-a$".
2. Show that $a^{-1} \cdot b$ is the solution of $a \cdot x = b$ and that $b \cdot a^{-1}$ is the solution of $y \cdot a = b$.
3. Which of the multiplication tables considered in Section 15-2 satisfy the group requirements? In case of failure, state the reason. In the case(s) where a group is specified, exhibit the identity element and the inverse of each element.
4. Let A denote a non-empty set, and $\cdot$ an operation in A. Show that there is at most one element $e \in A$ such that $a \cdot e = e \cdot a = a$ for all $a \in A$.

5. Let A denote a non-empty set, and $\cdot$ an operation in A. Suppose that $\cdot$ satisfies the associative law. Suppose that there exists an element $e \in A$ such that $a \cdot e = e \cdot a = a$ for all $a \in A$. (The element e is unique by Exercise 4.) Suppose that for each $a \in A$, there exists $x \in A$ such that $a \cdot x = e$ and that there exists $y \in A$ such that $y \cdot a = e$. Show that A together with $\cdot$ is a group. Hint: With x satisfying $a \cdot x = e$ and y satisfying $y \cdot a = e$, show that $a \cdot z = b$ is satisfied by $x \cdot b$, and, by multiplying each side by y, that the only possible solution is $y \cdot b$. Hence conclude that there is precisely one solution. Treat the remaining case similarly.
6. Construct multiplication tables for operations in a set A of three elements so that the group postulates G 1 and G 2 are satisfied. Hint: We may assume that one of the elements is e, the identity, and we may call one of the remaining elements a and the other b. The construction of a multiplication table can be carried out in only one way when account is taken of the nature of the identity element and the group postulates.

15-5. An Example of a Non-Abelian Group.

It is not hard to give an example of a group which is not abelian by means of a specifically constructed multiplication table. However, there is greater interest in constructing an example which is meaningful in terms of our earlier experience and which at the same time is important in terms of our future study of mathematics. The elements which we consider are the non-constant linear functions; that is, the functions f defined for all real numbers by the formulas of the form

$$f(x) = \alpha x + \beta ,$$

where α and β are real numbers and $\alpha \neq 0$. Our set A is taken to be the set whose elements are the functions f.

It should be observed that a given linear function is defined by precisely one formula of the form 15-5a. That is, if

$$\alpha x + \beta = \gamma x + \delta$$

for all real x, then $\alpha = \gamma$ and $\beta = \delta$. This is seen by first setting $x = 0$ and inferring that $\beta = \delta$ and then that $\alpha = \gamma$.

Composition. Suppose that we are given non-constant linear functions ℓ and m where $\ell(x) = \alpha x + \beta$ and $m(x) = \gamma x + \delta$. It is often of interest to construct a function from the given functions ℓ and m in the following manner. Starting with input x our first function ℓ yields output $\ell(x)$. Suppose that we now use $\ell(x)$ as input with the function m. The output is $m(\ell(x))$. We see that for each real x the quantity $m(\ell(x))$ is unambiguously specified. Thus we have a function determined by the requirement that to each real x there is assigned $m(\ell(x))$. This function is called the composition of m and ℓ. It is denoted by $m \bullet \ell$. Let us determine $m(\ell(x))$ explicitly. We have

$$\begin{aligned} m(\ell(x)) &= \gamma(\ell(x)) + \delta \\ &= \gamma(\alpha x + \beta) + \delta \\ &= \alpha\gamma x + (\beta\gamma + \delta). \end{aligned} \qquad \text{15-5b}$$

This computation shows that the function $m \circ \ell$ is a non-constant linear function, for the coefficient of x in the last line of Formula 15-5b is not zero. The rule which assigns to the ordered pair (m, ℓ) of non-constant linear functions the composition function $m \circ \ell$ is an operation in A. By analogy with what we did with sum and product, we denote the operation of composition by $\circ$. Let us pause to consider a numerical example before we continue our study of the structure we have just introduced.

Thus, suppose

$$\ell(x) = 2x + 1 \quad \text{and} \quad m(x) = -2x + 3 .$$

We have for $\ell \circ m$:

$$\ell(m(x)) = 2m(x) + 1 = 2(-2x + 3) + 1 = -4x + 7 .$$

We have for $m \circ \ell$:

$$M(\ell(x)) = -2\ell(x) + 3 = -2(2x + 1) + 3 = -4x + 1 .$$

This example shows that with the specific choices made for ℓ and m , we have

$$\ell \circ m \neq m \circ \ell .$$

We recall that two functions which have the same input sets (i.e., domain) are different if they assign different outputs for some member of their common input set. In our example $\ell \circ m$ and $m \circ \ell$ assign different outputs for *each* real x . Hence they are distinct functions.

This example shows us that the commutative law does *not* hold for the operation of composition of (non-constant) linear functions.

How do we show that the structure consisting of the non-constant linear functions together with the operation of composition is a group? We simply verify that G 1 and G 2 are fulfilled with the operation of composition.

G 1 . Suppose that ℓ, m, and n are three given (non-constant) linear functions. Given x as input, $\ell \circ (m \circ n)$ assigns as output *the ℓ output for input* $m \circ n(x)$, i.e., *the output for input* $m(n(x))$. Given x as input $(\ell \circ m) \circ n$ assigns as output *the $\ell \circ m$ output for input* $n(x)$, that is,

$$\ell \circ m(n(x)) .$$

But $\ell \circ m(n(x))$ is the ℓ output for input $m(n(x))$. Hence for each real x as input, $\ell \circ (m \circ n)$ and $(\ell \circ m) \circ n$ assign the same output. Hence the functions $\ell \circ (m \circ n)$ and $(\ell \circ m) \circ n$ are equal. The associative law G 1 is verified for composition.

G 2. Given two members of A, ℓ and m, we ask: Is there a member n satisfying

15-5c $$\ell \circ n = m \, ;$$

is there just one such member? Let us try to approach the question in an exploratory way. Let

$$\ell(x) = \alpha x + \beta , \quad m(x) = \gamma x + \delta .$$

Suppose that

$$n(x) = \lambda x + \mu \qquad (\lambda \neq 0)$$

satisfies 15-5c. From 15-5b we have

$$\ell \circ n(x) = \alpha \lambda x + (\beta + \alpha \mu) .$$

Hence if $\ell \circ n = m$, we have, using the fact that a linear function may be represented by only one formula of the form 15-5a,

$$\alpha \lambda = \gamma , \quad \beta + \alpha \mu = \delta .$$

Hence

15-5d $$\lambda = \gamma / \alpha , \quad \mu = \frac{\delta - \beta}{\alpha} .$$

We conclude that there is at most one such member n. On the other hand, if we take λ and μ as given by 15-5d, the function n defined by

$$n(x) = \lambda x + \mu$$

does satisfy 15-5c. Hence 15-5c has a unique solution.

The treatment of the other equation, $n \circ \ell = m$, where ℓ and m are given members of A, is similar. Thus we see that the set of non-constant linear functions together with the operation of composition is a non-abelian group.

Exercises 15-5

1. Furnish the details concerning the equation $n \circ \ell = m$, where ℓ and m are given members of A.
2. Determine the identity element of the group which we have studied in this section.
3. Determine the inverse of ℓ if $\ell(x) = \alpha x + \beta$, $\alpha \neq 0$.
4. Show by direct computation that $n = \ell^{-1} \circ m$ satisfies $\ell \circ n = m$ and that $n = m \circ \ell^{-1}$ satisfies $n \circ \ell = m$ where $\ell(x) = \alpha x + \beta$ and $m(x) = \gamma x + \delta$, $\alpha \neq 0$, $\gamma \neq 0$.
5. Show that $\ell \circ m = m \circ \ell$ for the functions of Exercise 4 if and only if $(\alpha - 1)\delta = (\gamma - 1)\beta$.
6. Let A denote the set of ordered pairs of real numbers with non-zero first components. Given (a,b), (c,d) in A, let $(a,b) \cdot (c,d)$ be defined as $(ac, ad + b)$. Show that $(A, \cdot)$ is a group. What is the identity element? What is the inverse of the element (a,b) of A? Is there any relation between this group and the group of non-constant linear functions treated in this section? Hint: Use No. 5 of Exercises 15-4.
7. Suppose that A is the set of ordered pairs of rational numbers with non-zero first components and that $\cdot$ is defined as in Exercise 6. Show that $(A, \cdot)$ is a group. Show that a corresponding result holds when A is the set of ordered pairs of complex numbers with non-zero first components and again $\cdot$ is defined as in Exercise 6.

15-6. Field.

We now turn to the consideration of an algebraic structure which is present in very many areas of mathematical study. We refer to the notion of a field. Once the definition of a field is stated, it will be clear that each of the following number systems is a field:

(a) The rationals with the usual addition and multiplication.

(b) The reals with the usual addition and multiplication.

(c) The complex numbers with the usual addition and multiplication.

Let A denote a set containing more than one member. Let $+$ and $\cdot$ denote two operations in A. Then $(A, +, \cdot)$ is called a field provided that the following postulates are satisfied:

F 1. The structure $(A, +)$ is an abelian group. (The identity element of this group is called "zero", and is denoted by "0" in accordance with the usage employed for the number systems which we have studied earlier; the inverse of the element a is denoted by $-a$, and the solution of $a + x = b$ by $b - a$).

F 2. Let B denote the set obtained from A by the removal of the element 0. It is required

(1) that $\cdot$ be an operation in B--i.e., if $b_1, b_2 \in B$, then $b_1 \cdot b_2 \in B$; and

(2) that the structure $(B, \cdot)$ be an abelian group. (The identity element of this group is called "one" and is denoted by "1". When we speak of $\cdot$ as an operation in B, we actually refer, not to the full operation $\cdot$ in A, but rather to the function obtained from $\cdot$ by restricting attention to inputs of the form (b_1, b_2) where b_1 and b_2 are members of B.)

F 3. The two distributive laws

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$
$$(b + c) \cdot a = b \cdot a + c \cdot a,$$

hold, a, b, and c being arbitrary elements of A.

Some remarks are in order.

Given a field $(A, +, \cdot)$, it is sometimes convenient in order to avoid unnecessarily clumsy modes of expression to use the phrase "the field A" and to mean either

(1) the set A, or

(2) the field in the strict sense $(A, +, \cdot)$.

Which meaning is intended will be clear from context. When we speak of the elements of the field, we mean of course the elements of A.

We shall also agree to write, as is usual, "ab" for "$a \cdot b$".

Of course, it is possible to state the required postulates in alternative form and in detail. The group concept, however, permits us to separate off in individual compartments a description of the action of each of the given operations $+$ and $\cdot$. It is now clear that if the two operations are to be interrelated in a serious sort of way, some condition pertaining to both $+$ and $\cdot$ must be in effect. In the postulates which we have listed, it is F 3 which links $+$ and $\cdot$. In particular, it is natural to turn to F 3 to see how 0 acts in multiplication.

We have

$$0 + 0 = 0 ,$$

and hence if a is an arbitrary element of A,

$$a(0 + 0) = a0 ,$$

and

$$(0 + 0)a = 0a .$$

Applying the distributive laws, we obtain

$$a0 + a0 = a0$$

and

$$0a + 0a = 0a ,$$

relations which state that $a0$ and $0a$ are each the zero of A; i.e.,

$$a0 = 0a = 0 , \qquad a \in A .$$

Postulate F 2 pertains only to B . Are the commutative and associative laws in effect for $\cdot$ in A ? The only case that need concern us is when one of the given elements is zero, but then we see that the two laws are in effect, for each side is zero if one of the given elements is.

Since $1 \cdot 0 = 0$ and $1 \cdot a = a$, $a \neq 0$, we see that 1 is an identity element for $\cdot$ in A . The element 1 is the only element in A with this property. If $e \in A$ satisfies $a \cdot e = a$ for all $a \in A$, we have

$$1 \cdot e = 1$$

and

$$1 \cdot e = e .$$

Hence

$$1 = e .$$

Consider equation $a \cdot x = b$. If $a = 0$ and $b \neq 0$, then there is no solution. If $a = 0$ and $b = 0$, then every element of A is a solution. Suppose that $a \neq 0$. Here we see, using the same argument that we used in the study of a group, that if $a \neq 0$, the equation has the unique solution $a^{-1} \cdot b$. Again, following our earlier practice for number systems, we shall denote the solution of $a \cdot x = b$, $a \neq 0$, by $\frac{b}{a}$.

We now see that the identities and theorems which were obtained for the rational number system, the real number system, or the complex number system, and whose proofs depended only on the structural laws which hold for an arbitrary field, continue to hold for an arbitrary field. Thus, if a, b, c, d are members of an arbitrary field and $b \neq 0$ and $d \neq 0$, then

15-6a $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} .$$

Exercises 15-6

1. Verify that Equation 15-6a holds for an arbitrary field.
2. Given that a, b, c, d are elements of a field and that $b \neq 0$, $c \neq 0$, $d \neq 0$. Show that $(a/b)/c = a/bc$ and that $(a/b)/(c/d) = ad/bc$.
3. Show that if a, b, c, d, e, f are arbitrary elements of a field and $ae - bd \neq 0$, then the system of equations

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

has a unique solution (x,y) whose components are elements of the field. Give explicit formulas for the solution.

4. Let A consist of the numbers 0, 1, 2. Let an operation $+$ be defined in A by the requirement that if $a, b \in A$, then $a + b$ is to be the remainder obtained when the number $a + b$ ($+$ being the conventional addition) is divided by 3. Thus if $a = 2$ and $b = 2$, then $a + b$ is the remainder obtained when $4 = 2 + 2$ is divided by 3; i.e., 1. Similarly, let an operation $\cdot$ be defined in A by the requirement that, if $a, b \in A$, then $a \cdot b$ is to be the remainder when the number ab (reference being made to conventional multiplication) is divided by 3. Display the tables for $+$ and $\cdot$. Verify that the structure $(A, +, \cdot)$ is a field. This exercise yields an example of a field which has precisely 3 **elements**.
5. Let A consist of two distinct elements a, b. Let $+$ and $\cdot$ be the operations in A given by the following tables.

+	a	b
a	a	b
b	b	a

·	a	b
a	a	a
b	a	b

Show that the structure (A, +, ·) is a field.

Specify the additive identity and the multiplicative identity of this field.

15-7. Subfield.

Given a field whose elements constitute a set A . It is natural to consider subsets B of A which taken together with + and · make up a field; that is, subsets B which have the following two properties:

(1) When + and · are restricted to ordered pairs (b_1,b_2) , whose components are in B , they define operations in B .

(2) B together with + and · so restricted is a field.

Such a subset B of A is called a subfield of A . Of course, one can also call such a B taken together with its two operations a subfield of the given field. The meaning which is intended will be clear from context.

With this notion we can proceed to find out something about the architecture of the complex number system. Let Q denote the set of rational numbers; let R denote the set of real numbers, and let C denote the set of complex numbers. We know that Q is a subset of R and that R is a subset of C ; in the notation of the theory of sets,

$$Q \subset R \subset C \ .$$

We may ask whether there are any intermediate subfields between R and C or between Q and R , and whether there is any subfield of the complex number system which is a proper part of Q .

Suppose that A is a subfield of the complex number system which contains R . Suppose that A contains an element not already in R . Then such an element must be of the form a + bi where a and b are real and $b \neq 0$. Since $a \in A$, $(a + bi) - a = bi \in A$. Since $b \in A$, $i \in A$. Hence given arbitrary real numbers c and d , we have $di \in A$ and therefore $c + di \in A$. That is, $C \subset A$. Hence $A = C$. We are led to the following conclusion:

Theorem 15-7a: If A is a subfield of the complex number system containing R , then either $A = R$ or $A = C$.

This theorem states that there is no subfield of the complex number system which contains R as a proper subset and at the same time is a proper subset of C .

A second result that is easy to obtain is the following:

Theorem 15-7b: Every subfield of the complex number system contains Q .

Proof: Let A denote a subfield of the complex number system. We note that if a and b belong to A and $b \neq 0$, then $\frac{a}{b} \in A$. Now $1 \in A$. It is a consequence of the additive closure of A and the well order property of the natural number system that every natural number is a member of A . Suppose that there are one or more natural numbers not in A and let m be the minimal member of the set of natural numbers not in A (the well order property assures us there is such a minimal member). Then $m - 1$ is a member of A , but our hypothesis tells us m is not. Since $m = (m - 1) + 1$ and $m - 1$ and 1 are in A , it follows from the additive closure of A that m itself is in A . This contradiction proves that the set of natural numbers not in A is empty. It now follows that every integer is a member of A , since for each natural number n , $-n$ is a member of A . Since A contains the quotients of its members, it follows that A contains every quotient of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. This says that every rational number is a member of A . In other words, $Q \subset A$. The theorem is established.

Subfields intermediate to Q and R . There is a vast hierarchy of subfields between Q and R . Their study is a large undertaking. We shall content ourselves to see that certain intermediate fields can be exhibited in a simple way.

Let A denote the set of real numbers of the form

$$a + b\sqrt{2}$$

where a and b are both rational. What can be said about sum and product of elements of A ? Given a, b, c, d rational, we see that

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} ,$$

and since $a + c$ and $b + d$ are rational, we have

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) \in A .$$

Similarly,

$$(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} ,$$

and since $ac + 2bd$ and $ad + bc$ are rational, we have

$$(a + b\sqrt{2})(c + d\sqrt{2}) \in A .$$

Suppose that $a + b\sqrt{2} = 0$ where a and b are rational. Then $b = 0$, otherwise $\sqrt{2}$ would be rational. It follows that also $a = 0$. Therefore, a member $a + b\sqrt{2}$ of A (a and b rational) is equal to zero if and only if $a = 0$ and $b = 0$. This implies that if $a + b\sqrt{2} \neq 0$, then $a^2 - 2b^2 \neq 0$. Otherwise we should have

$$0 = a^2 - 2b^2 = (a + b\sqrt{2})(a + (-b)\sqrt{2}) ,$$

so that either $a + b\sqrt{2} = 0$ or $a + (-b)\sqrt{2} = 0$. From $a + (-b)\sqrt{2} = 0$, we have $a = 0$ and $-b = 0$ and consequently

$a + b\sqrt{2} = 0$. That is, if $a^2 - 2b^2 = 0$, then $a + b\sqrt{2} = 0$.

We now have by a familiar rationalization method, for

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} = \frac{(a + b\sqrt{2})(c - d\sqrt{2})}{(c + d\sqrt{2})(c - d\sqrt{2})}$$

$$= \frac{(ac - 2bd)}{c^2 - 2d^2} + \frac{(bc - ad)\sqrt{2}}{c^2 - 2d^2} .$$

This tells us that the quotient of two members of **A** is also a member of A .

It is now easy to verify that A is a subfield of the real number system. We leave the details as an exercise.

Exercises 15-7

1. Show that A is a subfield of the real number system.
2. Let B denote the set of real numbers of the form $a + b\sqrt{3}$ where a and b are rational. Show that B is a subfield of the real number system.
*3. Show that the only real numbers belonging to both A and B are rational. In particular, $\sqrt{3}$ does not belong to A . Hence, A is *intermediate in the strict sense to* Q *and* R . That is, Q is a proper part of A, and A is a proper part of R.

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1. Birkhoff, Garrett and Saunders MacLane, *A Survey of Modern Algebra* (rev. ed.), Macmillan Company.
2. Books cited in the bibliography of Reference 1 above.

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